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Mathematical Problems in Mechanics

Asymptotic behavior of an elastic beam fixed on a small part of one of its extremities

Juan Casado-Díaz [a], Manuel Luna-Laynez [a], François Murat [b]

[a] *Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, c/ Tarfia s/n, 41012 Sevilla, Spain*
[b] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France*

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Abstract

We study the asymptotic behavior of the solution of an anisotropic, heterogeneous, linearized elasticity problem in a cylinder whose diameter ε tends to zero. The cylinder is assumed to be fixed (homogeneous Dirichlet boundary condition) on the whole of one of its extremities, but only on a small part (of size $\varepsilon r^\varepsilon$) of the second one; the Neumann boundary condition is imposed on the remainder of the boundary. We show that the result depends on r^ε, and that there are 3 critical sizes, namely $r^\varepsilon = \varepsilon^3$, $r^\varepsilon = \varepsilon$, and $r^\varepsilon = \varepsilon^{1/3}$, and in total 7 different regimes. We also prove a corrector result for each behavior of r^ε. ***To cite this article: J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Comportement asymptotique d'une poutre élastique fixée sur une petite partie de l'une de ses extrémités. Nous étudions le comportement asymptotique de la solution d'un problème d'élasticité linéaire anisotrope et hétérogène dans un cylindre dont le diamètre ε tend vers zéro. Le cylindre est fixé (condition de Dirichlet homogène) sur la totalité de l'une de ses extrémités, mais seulement sur une petite partie (de taille $\varepsilon r^\varepsilon$) de l'autre base ; sur le reste de la frontière on a la condition de Neumann. Nous montrons que le résultat depend de r^ε, et qu'il existe 3 tailles critiques, à savoir $r^\varepsilon = \varepsilon^3$, $r^\varepsilon = \varepsilon$ et $r^\varepsilon = \varepsilon^{1/3}$, et au total 7 comportements différents. Nous donnons un résultat de correcteur pour tous les comportements de r^ε. ***Pour citer cet article : J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Dans cette Note, nous étudions le comportement asymptotique de la solution d'un probleme d'élasticité linéaire anisotrope et hétérogène posé dans un cylindre $\Omega^\varepsilon = (0,1) \times \varepsilon S \subset \mathbb{R}^3$ dont le diamètre ε tend vers zéro et dont l'axe est le premier axe de coordonnées Ox_1. À l'une de ses extrémités ($x_1 = 1$) le cylindre est fixé (condition de Dirichlet homogène) sur la totalité de la base $\Gamma_1^\varepsilon = \{1\} \times \varepsilon S$, tandis qu'à l'autre extrémité ($x_1 = 0$), il est seulement fixé sur une petite partie $\Gamma_0^\varepsilon = \{0\} \times \varepsilon r^\varepsilon S_0$ de la base, partie qui est bien plus petite que ε car r^ε tend vers zéro.

E-mail addresses: jcasadod@us.es (J. Casado-Díaz), mllaynez@us.es (M. Luna-Laynez), murat@ann.jussieu.fr (F. Murat).

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Sur le reste de la frontière de Ω^ε, on a la condition de Neumann. Pour A tenseur d'ordre 4 coercif à coefficients continus dans $\overline{\Omega}$ ($\Omega = (0,1) \times S$), et $f \in L^2(\Omega)^3$, $h \in L^2(\Omega)_s^{3\times 3}$ donnés, on définit A^ε, F^ε et H^ε par (1) et (2) et on définit U^ε comme la solution du problème d'élasticité (3).

Le but de cette Note est de décrire le comportement asymptotique de U^ε et de donner un résultat de correcteur pour $e(U^\varepsilon)$ quand ε et r^ε tendent vers zéro. Ce résultat est décrit dans le théorème enoncé dans la version anglaise ci-dessous, qui fait apparaitre 3 tailles critiques, à savoir $r^\varepsilon \approx \varepsilon^3$, $r^\varepsilon \approx \varepsilon$ et $r^\varepsilon \approx \varepsilon^{1/3}$, qui séparent 4 zones correspondant à $r^\varepsilon \ll \varepsilon^3$, $\varepsilon^3 \ll r^\varepsilon \ll \varepsilon$, $\varepsilon \ll r^\varepsilon \ll \varepsilon^{1/3}$ et $\varepsilon^{1/3} \ll r^\varepsilon \leqslant C$, donc au total 7 cas différents. Le résultat de convergence est donné par (5), où u est défini par la solution de (4), tandis que (6) est un résultat de correcteur pour $e(U^\varepsilon)$.

1. Position of the problem and notation

In this Note we study the asymptotic behavior of the solution of an anisotropic, heterogeneous, linearized elasticity problem posed in a thin cylinder Ω^ε whose diameter ε tends to zero and whose axis is the first axis of coordinates Ox_1. On one of its extremities ($x_1 = 1$) the cylinder is fixed on its whole basis Γ_1^ε whereas on the second one ($x_1 = 0$) it is fixed only on a small part Γ_0^ε of it, of diameter $\varepsilon r^\varepsilon$ much smaller than ε. The Neumann boundary condition is imposed on the remainder of the boundary of Ω^ε. Mathematically the problem can be formulated as follows.

For $\varepsilon > 0$, we consider r^ε a positive parameter which tends to zero with ε. Let S_0 and S be two bounded smooth domains in $\mathbb{R}^2$, with $0 \in S$. We define $\Omega = (0,1) \times S$, $\Omega^\varepsilon = (0,1) \times \varepsilon S$ and $\Gamma^\varepsilon = \Gamma_0^\varepsilon \cup \Gamma_1^\varepsilon$, where $\Gamma_0^\varepsilon = \{0\} \times \varepsilon r^\varepsilon S_0$, $\Gamma_1^\varepsilon = \{1\} \times \varepsilon S$. Observe that the size of Γ_0^ε is much smaller than the size of the basis $\{0\} \times \varepsilon S$ since r^ε tends to zero.

The elements of $\mathbb{R}^3$ are decomposed as $x = (x_1, x')$, $x_1 \in \mathbb{R}$, $x' = (x_2, x_3) \in \mathbb{R}^2$. We denote by $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ the canonical basis of $\mathbb{R}^3$ and by $\mathcal{L}(\mathbb{R}_s^{3\times 3})$ the space of linear maps of $\mathbb{R}_s^{3\times 3}$ into itself (or in other terms of fourth order tensors), where $\mathbb{R}_s^{3\times 3}$ is the space of the 3×3 symmetric matrices. We adopt Einstein's convention of repeated indices. Greek indices (α and β) take only the values 2 and 3, while latin indices (i and j) take the values 1, 2 and 3.

We consider $A \in C^0(\overline{\Omega}; \mathcal{L}(\mathbb{R}_s^{3\times 3}))$ such that there exists $m > 0$ with $A(y)\xi\xi \geqslant m|\xi|^2$, for all $\xi \in \mathbb{R}_s^{3\times 3}$ and for all $y \in \overline{\Omega}$, and we define $A^\varepsilon \in C^0(\Omega^\varepsilon; \mathcal{L}(\mathbb{R}_s^{3\times 3}))$ by

$$A^\varepsilon(x) = A\left(x_1, \frac{x'}{\varepsilon}\right), \quad \forall x \in \Omega^\varepsilon. \tag{1}$$

We also consider $f \in L^2(\Omega)^3$ and $h \in L^2(\Omega)_s^{3\times 3}$, and we define $F^\varepsilon \in L^2(\Omega^\varepsilon)^3$ and $H^\varepsilon \in L^2(\Omega^\varepsilon)_s^{3\times 3}$ by

$$F^\varepsilon(x) = f_1\left(x_1, \frac{x'}{\varepsilon}\right)\mathbf{e}^1 + \varepsilon f_\alpha\left(x_1, \frac{x'}{\varepsilon}\right)\mathbf{e}^\alpha, \quad H^\varepsilon(x) = h\left(x_1, \frac{x'}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega^\varepsilon. \tag{2}$$

In the thin domain Ω^ε we consider the elasticity problem

$$\begin{cases} U^\varepsilon \in H^1_{\Gamma^\varepsilon}(\Omega^\varepsilon)^3, \\ \displaystyle\int_{\Omega^\varepsilon} A^\varepsilon e(U^\varepsilon) : e(\overline{U}^\varepsilon)\,\mathrm{d}x = \int_{\Omega^\varepsilon} F^\varepsilon \overline{U}^\varepsilon\,\mathrm{d}x + \int_{\Omega^\varepsilon} H^\varepsilon : e(\overline{U}^\varepsilon)\,\mathrm{d}x, \quad \forall \overline{U}^\varepsilon \in H^1_{\Gamma^\varepsilon}(\Omega^\varepsilon)^3, \end{cases} \tag{3}$$

where $H^1_{\Gamma^\varepsilon}(\Omega^\varepsilon) = \{U \in H^1(\Omega^\varepsilon)\colon U = 0 \text{ on } \Gamma^\varepsilon\}$; observe that in the above formulation, as well as in the remainder of the present Note, complex numbers never appear, and that $\overline{U}^\varepsilon$ (and later $\bar{u}$, $\bar{v}$, $\overline{w}$) denotes the test function associated to the solution U^ε, and not its complex conjugate. Observe also that the solution U^ε of (3) satisfies a nonhomogeneous Neumann boundary condition on the part $\partial\Omega^\varepsilon \setminus \Gamma^\varepsilon$ where it is not fixed, since integrating by parts $\int_{\Omega^\varepsilon} H^\varepsilon : e(U^\varepsilon)\,\mathrm{d}x$ (when h and therefore H^ε is sufficiently smooth) produces both body forces and surface forces. Similarly to the body forces F^ε we could have introduced explicit surface forces G^ε on $\partial\Omega^\varepsilon \setminus \Gamma^\varepsilon$, but we have preferred not to include them for the sake of simplicity.

It is well known that problem (3) has an unique solution (see, e.g., [5]). The aim of the present Note, which announces our paper [4], is to describe the asymptotic behavior of the solution U^ε and to give a corrector result for $e(U^\varepsilon)$ as ε tends to zero. The result depends on the behavior of r^ε and exhibits 3 critical sizes, namely ε^3, ε, and $\varepsilon^{1/3}$, so that there are 7 different regimes: $r^\varepsilon \ll \varepsilon^3$, $r^\varepsilon \approx \varepsilon^3$, $\varepsilon^3 \ll r^\varepsilon \ll \varepsilon$, $r^\varepsilon \approx \varepsilon$, $\varepsilon \ll r^\varepsilon \ll \varepsilon^{1/3}$, $r^\varepsilon \approx \varepsilon^{1/3}$, and $\varepsilon^{1/3} \ll r^\varepsilon \leqslant C$, where $r^\varepsilon \ll \varepsilon^\lambda$ stands for $r^\varepsilon/\varepsilon^\lambda \to 0$ (and equivalently $\varepsilon^\lambda \ll r^\varepsilon$ for $r^\varepsilon/\varepsilon^\lambda \to +\infty$), while $r^\varepsilon \approx \varepsilon^\lambda$ stands for $r^\varepsilon/\varepsilon^\lambda \to \rho$, for some ρ with $0 < \rho < +\infty$.

To express the results and to make the proofs, we will use two changes of variables.

The first change of variables is given by $y = y^\varepsilon(x)$ with $y_1 = x_1$, $y' = x'/\varepsilon$, which transforms the variable domain Ω^ε into the fixed domain Ω. This is the usual change of variables used to study equations in thin cylinders (see, e.g., [6,9–12]). When $r^\varepsilon = 1$ and $S_0 = S$ (but the same proof works for $r^\varepsilon = c$ independent of ε such that $cS_0 \subset S$), it was used successfully in [6,9–11] to pass to the limit in (3). When r^ε tends to zero with ε, this first change of variables allows us to describe the behavior of U^ε in the part of Ω^ε far from Γ_0^ε (this behavior is the same as in the case where $r^\varepsilon = c$), but it does not provide the information we need about the behavior of U^ε in the part of Ω^ε close to Γ_0^ε.

Thus we introduce a second change of variables given by $z = z^\varepsilon(x)$ with $z = x/\varepsilon r^\varepsilon$, which transforms the variable domain Ω^ε into a variable domain Z^ε, the limit of which is the half space $Z = (0, +\infty) \times \mathbb{R}^2$. Observe that the Dirichlet boundary condition is now imposed on the fixed part $\{0\} \times S_0$ of the boundary of Z^ε. This change of variables provides a suitable rescaling near $x_1 = 0$. It was used successfully in [2,3] to study the diffusion problem similar to (3) in the geometry that we consider in the present Note, and even in a more complicated one (see also [1]).

We denote by $D^{1,2}(Z)$ the Deny's space $D^{1,2}(Z) = \{p\colon\ p \in L^6(Z),\ \nabla p \in L^2(Z)^3\}$. We will also use the functional spaces (already used in [7,8,10,11])

$$
\begin{cases}
BN_b(\Omega) = \Big\{u\colon\ \exists \zeta_\alpha \in H^2(0,1),\ \zeta_\alpha(1) = \dfrac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(1) = 0,\ u_\alpha(y) = \zeta_\alpha(y_1),\ \forall \alpha \in \{2,3\}, \\
\qquad\qquad \exists \zeta_1 \in H^1(0,1),\ \zeta_1(1) = 0,\ u_1(y) = \zeta_1(y) - \dfrac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(y_1) y_\alpha \Big\},
\end{cases}
$$

$$
\begin{cases}
R_b(\Omega) = \Big\{v\colon\ v_1 \in L^2\big(0,1; H^1(S)\big),\ \displaystyle\int_S v_1(y_1, y')\,\mathrm{d}y' = 0 \text{ a.e. } y_1 \in (0,1), \\
\qquad\qquad \exists c \in H^1(0,1),\ c(1) = 0,\ v_2(y) = c(y_1) y_3,\ v_3(y) = -c(y_1) y_2 \Big\},
\end{cases}
$$

$$
\begin{cases}
RD_2^\perp(\Omega) = \Big\{w\colon\ w_1 = 0,\ w_\alpha \in L^2\big(0,1; H^1(S)\big),\ \displaystyle\int_S w_\alpha(y_1, y')\,\mathrm{d}y' = 0,\ \forall \alpha \in \{2,3\}, \\
\qquad\qquad \displaystyle\int_S \big(y_3 w_2(y_1, y') - y_2 w_3(y_1, y')\big)\,\mathrm{d}y' = 0 \text{ a.e. } y_1 \in (0,1) \Big\}.
\end{cases}
$$

For a given $(u, v, w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega)$, we denote by $E(u, v, w)$ the second order symmetric tensor with values in $\mathbb{R}_s^{3\times 3}$ defined by $E_{11}(u, v, w) = e_{11}(u)$, $E_{1\beta}(u, v, w) = e_{1\beta}(v)$, $E_{\alpha\beta}(u, v, w) = e_{\alpha\beta}(w)$, $\forall \alpha, \beta \in \{2,3\}$.

2. The result and some comments

The asymptotic behavior of the solution of (3) depends on the size of r^ε with respect to ε. Seven regimes appear in the following theorem which describes the asymptotic behavior of U^ε and provides a corrector result for U^ε and $e(U^\varepsilon)$.

Theorem 2.1. *Let U^ε be the solution of* (3). *There exist a closed linear subspace $\mathcal{E}$ of $BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega)$, a function $P^\varepsilon \in L^2(\Omega^\varepsilon)_s^{3\times 3}$, and a bilinear continuous nonnegative form $\mathcal{B}$ on $\mathcal{E} \times \mathcal{E}$ such that, defining (u, v, w) as the solution of the variational problem*

$$\begin{cases} (u,v,w)\in\mathcal{E}, \\ \displaystyle\int_\Omega AE(u,v,w):E(\bar u,\bar v,\overline{w})\,\mathrm{d}y+\mathcal{B}\big((u,v,w),(\bar u,\bar v,\overline{w})\big)=\int_\Omega f\bar u\,\mathrm{d}y+\int_\Omega h:E(\bar u,\bar v,\overline{w})\,\mathrm{d}y, \\ \forall(\bar u,\bar v,\overline{w})\in\mathcal{E}, \end{cases} \tag{4}$$

then, when ε tends to zero, we have

$$\frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon}\left(\left|U_1^\varepsilon(x)-u_1\left(x_1,\frac{x'}{\varepsilon}\right)\right|^2+\sum_{\alpha=2}^{3}\left|\varepsilon U_\alpha^\varepsilon(x)-u_\alpha(x_1)\right|^2\right)\mathrm{d}x\longrightarrow 0, \tag{5}$$

$$\frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon}\left|e\big(U^\varepsilon\big)(x)-E(u,v,w)\left(x_1,\frac{x'}{\varepsilon}\right)-P^\varepsilon\left(\frac{x}{\varepsilon r^\varepsilon}\right)\right|^2\mathrm{d}x\longrightarrow 0. \tag{6}$$

The definitions of $\mathcal{E}$, P^ε and $\mathcal{B}$ do not depend on the forces f and h which define F^ε and H^ε, but only on the set S_0, on the fourth order tensor A, and on the behavior of r^ε when ε tends to zero, and more specifically of its behavior with respect to ε^3, ε, and $\varepsilon^{1/3}$, such that there are 7 regimes, which are described now.

- *If $r^\varepsilon\ll\varepsilon^3$, then $\mathcal{E}=BN_b(\Omega)\times R_b(\Omega)\times RD_2^\perp(\Omega)$, $P^\varepsilon=0$, $\mathcal{B}=0$.*
- *If $r^\varepsilon\approx\varepsilon^3$ with $r^\varepsilon/\varepsilon^3\to\rho$, then $\mathcal{E}=BN_b(\Omega)\times R_b(\Omega)\times RD_2^\perp(\Omega)$, and defining φ^i, $i\in\{1,2,3\}$, as the solution of*

$$\begin{cases} \varphi^i\in D^{1,2}(Z)^3, \quad \varphi^i=\mathbf{e}^i \quad \text{on } \{0\}\times S_0, \\ \displaystyle\int_Z A(0)e\big(\varphi^i\big):e(\bar\varphi)\,\mathrm{d}z=0, \quad \forall\bar\varphi\in D^{1,2}(Z)^3, \quad \bar\varphi=0 \quad \text{on } \{0\}\times S_0, \end{cases} \tag{7}$$

  then one has

$$P^\varepsilon(z)=-\frac{1}{\varepsilon^2 r^\varepsilon}\zeta_\alpha(0)\,e\big(\varphi^\alpha\big)(z), \quad \text{a.e. } z\in Z,$$

$$\mathcal{B}\big((u,v,w),(\bar u,\bar v,\overline{w})\big)=\rho\int_Z A(0)\big(\zeta_\alpha(0)\,e\big(\varphi^\alpha\big)\big):\big(\bar\zeta_\beta(0)\,e\big(\varphi^\beta\big)\big)\,\mathrm{d}z, \quad \forall(u,v,w),(\bar u,\bar v,\overline{w})\in\mathcal{E}.$$

- *If $\varepsilon^3\ll r^\varepsilon\ll\varepsilon$, then*

$$\mathcal{E}=\big\{(u,v,w)\in BN_b(\Omega)\times R_b(\Omega)\times RD_2^\perp(\Omega):\ \zeta_\alpha(0)=0,\ \forall\alpha\in\{2,3\}\big\}, \quad P^\varepsilon=0, \quad \mathcal{B}=0.$$

- *If $r^\varepsilon\approx\varepsilon$ with $r^\varepsilon/\varepsilon\to\rho$, then*

$$\mathcal{E}=\big\{(u,v,w)\in BN_b(\Omega)\times R_b(\Omega)\times RD_2^\perp(\Omega):\ \zeta_\alpha(0)=0,\ \forall\alpha\in\{2,3\}\big\},$$

  and setting $\hat\varphi^1=\varphi^1+a_\alpha\varphi^\alpha$, where φ^i is defined by (7) and where (a_2,a_3) is defined by

$$(a_2,a_3)\in\mathbb{R}^2, \quad \int_Z A(0)\big(e\big(\varphi^1\big)+a_\alpha\,e\big(\varphi^\alpha\big)\big):\big(\bar a_\beta\,e\big(\varphi^\beta\big)\big)\,\mathrm{d}z=0, \quad \forall(\bar a_2,\bar a_3)\in\mathbb{R}^2,$$

  then one has

$$P^\varepsilon(z)=-\frac{1}{\varepsilon r^\varepsilon}\,\zeta_1(0)e\big(\hat\varphi^1\big)(z), \quad \text{a.e. } z\in Z,$$

$$\mathcal{B}\big((u,v,w),(\bar u,\bar v,\overline{w})\big)=\rho\int_Z A(0)\big(\zeta_1(0)e\big(\hat\varphi^1\big)\big):\big(\bar\zeta_1(0)e\big(\hat\varphi^1\big)\big)\,\mathrm{d}z, \quad \forall(u,v,w),(\bar u,\bar v,\overline{w})\in\mathcal{E}.$$

- *If* $\varepsilon \ll r^\varepsilon \ll \varepsilon^{1/3}$, *then*

$$\mathcal{E} = \big\{(u,v,w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^{\perp}(\Omega):\ \zeta_\alpha(0) = \zeta_1(0) = 0,\ \forall \alpha \in \{2,3\}\big\}, \quad P^\varepsilon = 0, \quad \mathcal{B} = 0.$$

- *If* $r^\varepsilon \approx \varepsilon^{1/3}$ *with* $(r^\varepsilon)^3/\varepsilon \to \rho$, *then*

$$\mathcal{E} = \big\{(u,v,w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^{\perp}(\Omega):\ \zeta_\alpha(0) = \zeta_1(0) = 0,\ \forall \alpha \in \{2,3\}\big\},$$

and defining ψ^1 *as the solution of*

$$\begin{cases} \psi^1 \in D^{1,2}(Z)^3, \quad \psi^1 = z_3\mathbf{e}^2 - z_2\mathbf{e}^3 \quad \text{on } \{0\} \times S_0, \\ \displaystyle\int_Z A(0)e(\psi^1) : e(\bar\psi)\,\mathrm{d}z = 0, \quad \forall \bar\psi \in D^{1,2}(Z)^3, \quad \bar\psi = 0 \quad \text{on } \{0\} \times S_0, \end{cases}$$

and $\psi^\alpha,\ \alpha \in \{2,3\}$, *as the solution of*

$$\begin{cases} \psi^\alpha \in D^{1,2}(Z)^3, \quad \psi^\alpha = z_1\mathbf{e}^\alpha - z_\alpha\mathbf{e}^1 \quad \text{on } \{0\} \times S_0, \\ \displaystyle\int_Z A(0)e(\psi^\alpha) : e(\bar\psi)\,\mathrm{d}z = 0, \quad \forall \bar\psi \in D^{1,2}(Z)^3, \quad \bar\psi = 0 \quad \text{on } \{0\} \times S_0, \end{cases}$$

and then setting $\hat\psi^i = \psi^i + b^i_k \varphi^k$, *where* φ^k *is defined by* (7) *and where* $(b^i_1, b^i_2, b^i_3),\ i \in \{1,2,3\}$, *is defined by*

$$\big(b^i_1, b^i_2, b^i_3\big) \in \mathbb{R}^3, \quad \int_Z A(0)\big(e(\psi^i) + b^i_k e(\varphi^k)\big) : \big(\bar b^i_l e(\varphi^l)\big)\,\mathrm{d}z = 0, \quad \forall \big(\bar b^i_1, \bar b^i_2, \bar b^i_3\big) \in \mathbb{R}^3,$$

then one has

$$P^\varepsilon(z) = -\frac{1}{\varepsilon}\left(c(0)\,e\big(\hat\psi^1\big)(z) + \frac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(0)e\big(\hat\psi^\alpha\big)(z)\right), \quad \text{a.e. } z \in Z,$$

$$\begin{cases} \forall (u,v,w), (\bar u, \bar v, \overline{w}) \in \mathcal{E}, \quad \mathcal{B}\big((u,v,w), (\bar u, \bar v, \overline{w})\big) \\ \displaystyle\quad = \rho \int_Z A(0)\left(c(0)e\big(\hat\psi^1\big) + \frac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(0)e\big(\hat\psi^\alpha\big)\right) : \left(\bar c(0)e\big(\hat\psi^1\big) + \frac{\mathrm{d}\bar\zeta_\alpha}{\mathrm{d}y_1}(0)e\big(\hat\psi^\alpha\big)\right)\mathrm{d}z. \end{cases}$$

- *If* $\varepsilon^{1/3} \ll r^\varepsilon \leqslant C$, *then*

$$\mathcal{E} = \left\{(u,v,w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^{\perp}(\Omega):\ \zeta_\alpha(0) = \zeta_1(0) = \frac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(0) = c(0) = 0, \quad \forall \alpha \in \{2,3\}\right\},$$
$$P^\varepsilon = 0, \quad \mathcal{B} = 0.$$

Let us make some comments about the statement of this theorem.

Assertion (6) is a corrector result, since one can prove that $\frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon} |e(U^\varepsilon)|^2\,\mathrm{d}x + \frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon} |E(u,v,w) \times (x_1, \frac{x'}{\varepsilon})|^2\,\mathrm{d}x + \frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon} |P^\varepsilon(\frac{x}{\varepsilon r^\varepsilon})|^2\,\mathrm{d}x \approx 1$.

If one examines the definition of $\mathcal{E}$, one realizes that the number of Dirichlet boundary conditions imposed in the definition of $\mathcal{E}$ increases with the size of r^ε. Indeed, in view of the definitions of $BN_b(\Omega)$, $R_b(\Omega)$ and $RD_2^{\perp}(\Omega)$, the sole functions which have a trace for $x_1 = 0$ are ζ_α, ζ_1, $\frac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}$, and c, where $\alpha \in \{2,3\}$ (the other functions, namely v_1 and w_α, have no trace for $x_1 = 0$). When $r^\varepsilon \ll \varepsilon^3$, the set $\Gamma_0^\varepsilon = \{0\} \times \varepsilon r^\varepsilon S_0$ is too small and the homogeneous Dirichlet boundary condition imposed for $x_1 = 0$ to the solution U^ε of (3) completely disappears at the limit. When $\varepsilon^3 \ll r^\varepsilon \ll \varepsilon$, the set Γ_0^ε is sufficiently large to impose at the limit that $\zeta_\alpha(0) = 0$ for $\alpha \in \{2,3\}$. When $\varepsilon \ll r^\varepsilon \ll \varepsilon^{1/3}$, one further has $\zeta_1(0) = 0$. Finally when $\varepsilon^{1/3} \ll r^\varepsilon$, the set Γ_0^ε is so large that all the possible Dirichlet boundary conditions are imposed at $x_1 = 0$, namely $\zeta_\alpha(0) = \zeta_1(0) = \frac{\mathrm{d}\zeta_\alpha}{\mathrm{d}y_1}(0) = c(0) = 0$.

Except in the three regimes where the size of r^ε is critical (i.e., when $r^\varepsilon \approx \varepsilon^\lambda$, with $\lambda = 3,\ 1$, or $1/3$), one always has $P^\varepsilon = 0$ and $\mathcal{B} = 0$. For these three critical sizes, one can show that $\mathcal{B}$ is a coercive bilinear form, in the sense that there exists some $n > 0$ such that $\mathcal{B}((u,v,w),(u,v,w))$ is greater than $n\rho(|\zeta_2(0)|^2 + |\zeta_3(0)|^2)$ when $r^\varepsilon \approx \varepsilon^3$, than $n\rho|\zeta_1(0)|^2$ when $r^\varepsilon \approx \varepsilon$, and than $n\rho(|c(0)|^2 + |\frac{d\zeta_2}{dy_1}(0)|^2 + |\frac{d\zeta_3}{dy_1}(0)|^2)$ when $r^\varepsilon \approx \varepsilon^{1/3}$. This implies that for every critical size $r^\varepsilon \approx \varepsilon^\lambda$, the new Dirichlet boundary conditions which appear for $r^\varepsilon \gg \varepsilon^\lambda$ (with respect to those imposed for $r^\varepsilon \ll \varepsilon^\lambda$) are penalized by the value of ρ. This introduces some type of continuity in the transition of the Dirichlet condition between the two regimes which are separated by a critical size ε^λ. For these three critical sizes, the functions $\varphi^\alpha,\ \varphi^1,\ \hat{\varphi}^1,\ \psi^i$, and $\hat{\psi}^i$ are in some sense generalized capacitary potentials of $\{0\} \times S_0$ in Z, and the bilinear form $\mathcal{B}$ is in some sense an asymptotic trace of some type of capacity of Γ_0^ε in Ω^ε for the weighted energy $\frac{1}{|\Omega^\varepsilon|}\int_{\Omega^\varepsilon} A(x)e(\varphi) : e(\varphi)\,dx$.

The present work is the natural generalization to the elastic case of [2,3], where diffusion problems were posed in the union of two cylinders $\{(-t^\varepsilon, 0) \times \varepsilon r^\varepsilon S_0\} \cup \{(0,1) \times \varepsilon S\}$, with both t^ε and r^ε tending to zero (the present geometry corresponds to $t^\varepsilon = 0$). When $t^\varepsilon = 0$, the diffusion problem was in comparison more simple, since only one critical size, namely $r^\varepsilon \approx \varepsilon$, appeared in the analysis, separating the Neumann boundary condition (corresponding to the analogue of u satisfying $u \in H^1(0,1),\ u(1) = 0$) for $r^\varepsilon \ll \varepsilon$, and the Dirichlet boundary condition (corresponding to the analogue of u satisfying $u \in H^1(0,1),\ u(0) = u(1) = 0$) for $r^\varepsilon \gg \varepsilon$. These works were related to [1], where a notched beam for diffusion problems was considered. The present work is also related to [7,8], where a multidomain made of an elastic vertical beam of length 1 and of radius r^ε and of an horizontal plate of radius 1 and of height ε was considered. Finally, the present work is also the generalization of [10,11], which were concerned with the case $r^\varepsilon = c$, to the present case where r^ε tends to zero.

The detailed proofs of the results of the present Note will be given in a forthcoming paper [4].

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References

[1] J. Casado-Díaz, F. Murat, The diffusion equation in a notched beam, in preparation.
[2] J. Casado-Díaz, M. Luna-Laynez, F. Murat, Asymptotic behavior of diffusion problems in a domain made of two cylinders of different diameters and lengths, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 133–138.
[3] J. Casado-Díaz, M. Luna-Laynez, F. Murat, Diffusion problems in a domain made of two thin cylinders of different diameters and lengths, in preparation.
[4] J. Casado-Díaz, M. Luna-Laynez, F. Murat, Elasticity problems in a beam fixed on only a small part of one of its extremities, in preparation.
[5] P.G. Ciarlet, Mathematical Elasticity, vol. I: Three-Dimensional Elasticity, North-Holland, 1988.
[6] A. Cimetière, G. Geymonat, H. Le Dret, A. Raoult, Z. Tutek, Asymptotic theory and analysis for displacements and stress distribution in nonlinear straight slender rods, J. Elasticity 19 (1988) 111–161.
[7] A. Gaudiello, R. Monneau, J. Mossino, F. Murat, A. Sili, On the junction of elastic plates and beams, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 717–722.
[8] A. Gaudiello, R. Monneau, J. Mossino, F. Murat, A. Sili, Junction of elastic plates and beams, in press.
[9] H. Le Dret, Convergence of displacements and stresses in linearly elastic slender rods as the thickness goes to zero, Asymptotic Anal. 10 (1995) 367–402.
[10] F. Murat, A. Sili, Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces, C. R. Acad. Sci. Paris, Ser. I 328 (1999) 179–184.
[11] F. Murat, A. Sili, Anisotropic, heterogeneous, linearized elasticity in thin cylinders, in preparation.
[12] L. Trabucho, J.M. Viaño, Mathematical Modelling of Rods, in: Handbook of Numerical Analysis, vol. IV, North-Holland, 1996.