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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 751–753
COMPTES RENDUS MATHEMATIQUE

Number Theory

A counterexample to the Gouvêa–Mazur conjecture

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Received 4 March 2004; accepted 16 March 2004

Presented by Christophe Soulé

Abstract

Gouvêa and Mazur made a precise conjecture about slopes of modular forms. Weaker versions of this conjecture were established by Coleman and Wan. In this Note, we exhibit examples contradicting the full conjecture as it currently stands. ***To cite this article: K. Buzzard, F. Calegari, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Un contre-exemple à la conjecture de Gouvêa–Mazur. Gouvêa et Mazur ont proposé une conjecture précise au sujet des pentes des formes modulaires. Des versions plus faibles de cette conjecture ont été prouvées par Coleman et Wan. Dans cette Note, nous exhibons des exemples contredisant la conjecture. ***Pour citer cet article : K. Buzzard, F. Calegari, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Introduction – the Gouvêa–Mazur conjecture

Let p be a prime number, and let N be a positive integer coprime to p. For an integer k, let $f_k \in \mathbf{Z}[X]$ denote the characteristic polynomial of the Hecke operator U_p on the space of cusp forms of level $\Gamma_0(Np)$ and weight k. Normalise the multiplicative valuation $v:\overline{\mathbf{Q}}_p^\times \to \mathbf{Q}$ so that $v(p) = 1$. If $\alpha \in \mathbf{Q}$ then let $d(k,\alpha)$ denote the number of roots of f_k in $\overline{\mathbf{Q}}_p$ which have valuation equal to α.

Conjecture 1.1 (Gouvêa and Mazur, [5]). *If $k_1, k_2 \in \mathbf{Z}$ are both at least $2\alpha + 2$ and $k_1 \equiv k_2 \bmod p^n(p-1)$ for some integer $n \geqslant \alpha$, then $d(k_1,\alpha) = d(k_2,\alpha)$.*

This important conjecture was, as far as we know, one of the first precise formulations of the idea that Hida's theory of ordinary families might be generalised to the non-ordinary case. The conjecture can be thought of as saying that non-ordinary families of cuspidal eigenforms exist, and Gouvêa and Mazur went on to conjecture that

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[1] The second author is supported in part by the American Institute of Mathematics.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.03.016

these conjectural families should in fact be analytic in the weight variable. Many of the conjectures that Gouvêa and Mazur made in this area became theorems soon afterwards, as a result of deep ideas of Coleman [4], but the conjecture above, which we shall refer to as the Gouvêa–Mazur conjecture, remained open. Coleman's ideas, extended by Wan in [7], could only prove that if k_1 and k_2 were congruent modulo $p^N(p-1)$ for some integer $N = O(\alpha^2)$ then $d(k_1,\alpha) = d(k_2,\alpha)$. For a while the situation was almost paradoxical, because computations of Gouvêa, Stein, and the first author all seemed to indicate that in fact the Gouvêa–Mazur conjecture was much too weak. See for example [3] which introduces the notion of a prime p being $\Gamma_0(N)$-regular and makes some much more precise conjectures when this condition holds.

2. Counterexamples

If $N = 1$ then the smallest prime which is not $\Gamma_0(N)$-regular is $p = 59$, and the authors decided to attempt to study this case in detail. In particular, they wanted to study the point of the eigencurve corresponding to the slope 1 form of level 59 and weight 16. To their surprise, they discovered:

Theorem 2.1. *Set $N = 1$ and $p = 59$. Then there exists $\alpha \in \mathbf{Q}$ with $0 \leqslant \alpha \leqslant 1$ such that $d(16,\alpha) \neq d(3438,\alpha)$. In particular, because $3438 = 16 + 58 \cdot 59$, the Gouvêa–Mazur conjecture is false.*

Remark 1. We know $d(16,1) = 1$, and it is probably true that $d(3438,1) = 2$, but it is, at the present time, difficult to compute characteristic polynomials of Hecke operators at such high weight.

Proof. It is well known that the space of level 1 cusp forms of weight 16 is 1-dimensional, and that the 59-adic valuation of the eigenvalue of T_{59} is 1 on this space. Because any newforms of level 59 and weight 16 will have slope $(16-2)/2 = 7$, we see that $d(16,1) = 1$ and $d(16,\alpha) = 0$ for $0 \leqslant \alpha < 1$.

Now using the Eichler–Selberg trace formula (see for example Theorem 2 on p. 48 of [6], and [8]), it is possible to compute the trace of T_{59} and T_{59^2} on the space of cusp forms of level 1 and weight 3438. These can be computed as real numbers to very large precision in about 30 seconds using `pari-GP` on a 300 MHz Pentium III (the trace of T_{59^2} is about $6.79926\ldots \times 10^{6086}$), and these real numbers turn out to be very close to integers, so one can round to get the exact result. In fact, for the argument below, we only need to know the traces modulo 59^3 and these can be computed in less than a tenth of a second using a modular implementation of the trace formula.

Let the characteristic polynomial of T_{59} on the space of cusp forms of level 1 be $X^{286} + \sum_{i=1}^{286} a_i X^{286-i}$. From the above calculations one has enough data to compute a_1 and a_2 exactly (although we only need them modulo 59^3). One now checks that the 59-adic valuations of a_1 and a_2 are 1 and 2 respectively. There are now two cases to consider: either $v(a_i) \geqslant i$ for all i, in which case $d(3438,1) \geqslant 2$, or there exists i with $v(a_i) < i$, in which case $d(3438,\alpha) > 0$ for some α with $0 \leqslant \alpha < 1$. In either case the conjecture is violated. □

Remark 2. The authors strongly suspect that $v(a_i) \geqslant i$ for all i, and perhaps a more intense computation and some estimates on the a_i might prove this.

This counterexample comes from a point on an eigencurve with integral slope, whose associated mod p representation is irreducible even when restricted to a decomposition group at p. In fact a search for other similar points led to other counterexamples:

Theorem 2.2. *Set $p = 5$. Then the set of N with $1 \leqslant N \leqslant 83$ and such that $5 \nmid N$ and $d(6,1) > 0$ is $\{14, 28, 34, 37, 38, 42, 53, 56, 68, 69, 71, 74, 76, 83\}$, and for each N in this set we have $d(26,1) = 2d(6,1) > d(6,1)$, so every such N gives a counterexample to the Gouvêa–Mazur conjecture.*

Proof. One verifies this statement easily using William Stein's programs for modular forms, written for the MAGMA computer algebra package [1]. □

Remark 3. As opposed to the $p = 59$ counterexample above, which relies on "custom computations" done by the authors, these latter counterexamples rely only on standard programs of Stein which have existed for many years and are likely to be bug-free.

The counterexamples exhibited above all pertain to forms of positive slope and weight at most $p + 1$, and hence to forms whose associated mod p representation is irreducible when restricted to a decomposition group at p. After our counterexamples were discovered, Graham Herrick pointed out to us that if one extends the domain of the conjecture to forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ then one can find counterexamples coming from forms whose associated mod p representation is reducible, even globally. For example if $p = 5$ then there are two slope 1 forms of level $\Gamma_1(4)$ at weight 7, and four slope 1 forms of this level at weight 27. The mod 5 representations associated to all of these forms are reducible. Note that once again there are twice as many slope 1 forms in the higher weight than are predicted by the Gouvêa–Mazur conjecture.

Finally, we remark that Breuil has made local conjectures (Conjecture 1.5 in [2]) which imply that if f is an eigenform of level N and weight k with $2 \leqslant k \leqslant 2p$, and if the slope of f is strictly between 0 and 1, then the mod p representation associated to f is irreducible when restricted to a decomposition group at p. Breuil has asked (personal communication with K.B.) whether one should expect such a result when $k > 2p$. However, the mod 5 representations associated to the four forms of level $\Gamma_1(4)$ and weight 11 are all reducible, and all of these forms have slope $1/2$. The generalisation of Breuil's conjectures to the case $k > 2p$ remains an interesting open problem, which might shed more light on the results of this paper.

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