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String equation and quantum cohomology for noncompact symplectic manifolds

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Abstract

We use our Gromov–Witten invariant theory in a previous Note for noncompact geometrically bounded symplectic manifolds to get solutions of the generalized string equation and dilation equation and their variants. More solutions of the WDVV equation and quantum products on cohomology groups are also obtained for the symplectic manifolds with finitely dimensional cohomology groups. ***To cite this article: G. Lu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

L'équation des corde et cohomologie quantique pour variétés symplectiques non compactes. En utilisant la théorie des invariants de Gromov–Witten dans une Note précédente pour variétés symplectiques non compactes, géométriquement bornées, on obtient des solutions de l'équation généralisée des cordes, de l'équation de dilatation et de leurs variantes. On obtient également davantage de solutions de l'équation WDVV et des produits de quantum sur les groupes de cohomologie, pour les variétés symplectiques dont les groupes de cohomologie sont de dimension finie. ***Pour citer cet article : G. Lu, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. String equation

This is a continuation of our previous Note [1]. As expected our Gromov–Witten invariant theory for noncompact geometrically bounded symplectic manifolds can be used to get solutions of the generalized string equation and dilation equation and their variants. We follow the notations in [1] and [2] without special statements.

Let $\overline{\mathcal{U}}_{g,m}$ be the universal curve over $\overline{\mathcal{M}}_{g,m}$. The i-th marked point z_i yields a section $\tilde{z}_i$ of the fibration $\overline{\mathcal{U}}_{g,m} \to \overline{\mathcal{M}}_{g,m}$. Denote by $\mathcal{K}_{\mathcal{U}|\mathcal{M}}$ the cotangent bundle to fibers of this fibration, and $\mathcal{L}_i = \tilde{z}_i(\mathcal{K}_{\mathcal{U}|\mathcal{M}})$. For

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nonnegative integers $d_1, \ldots, d_m$ we also denote by $\kappa_{d_1,\ldots,d_m}$ the Poincaré dual of $c_1(\mathcal{L}_1)^{d_1} \cup \cdots \cup c_1(\mathcal{L}_m)^{d_m}$. We define (or make conventions)

$$\begin{cases} \mathcal{GW}^{(\omega,\mu,J)}_{A,g,m}(\kappa_{d_1,\ldots,d_m}; \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{m \text{ times}}) = 0, \quad \mathcal{GW}^{(\omega,\mu,J)}_{A,0,3}\big([\bar{\mathcal{M}}_{0,3}]; \mathbf{1},\mathbf{1},\mathbf{1}\big) = 0, \\ \mathcal{GW}^{(\omega,\mu,J)}_{A,1,1}\big([\bar{\mathcal{M}}_{1,1}]; \mathbf{1}\big) = 0, \quad \mathcal{GW}^{(\omega,\mu,J)}_{0,1,1}(\kappa_1; \mathbf{1}) = 0, \quad \mathcal{GW}^{(\omega,\mu,J)}_{0,1,1}\big([pt]; \mathbf{1}\big) = \chi(M) \end{cases} \tag{1}$$

because these cannot be included in the category of our definition. Here $\chi(M)$ is the Euler characteristic of M. Given an integer $g \geqslant 0$ and $A \in H_2(M)$, let one class in $\{\alpha_i\}_{1\leqslant i\leqslant m} \subset H^*_c(M,\mathbb{Q}) \cup H^*(M,\mathbb{Q})$ belong to $H^*_c(M,\mathbb{Q})$. Following [5] we call the invariant

$$\langle \tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_m,\alpha_m}\rangle_{g,A} = \mathcal{GW}^{(\omega,\mu,J)}_{A,g,m}(\kappa_{d_1,\ldots,d_m}; \alpha_1,\ldots,\alpha_m)$$

a *gravitational correlator*. The m-point genus-g correlators are defined by

$$\langle \tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_m,\alpha_m}\rangle_{g}(q) = \sum_{A\in H_2(M)} \mathcal{GW}^{(\omega,\mu,J)}_{A,g,m}(\kappa_{d_1,\ldots,d_m}; \alpha_1,\ldots,\alpha_m) q^A, \tag{2}$$

where q is an element of Novikov ring as before.

Note that for noncompact M, $H^0_c(M,\mathbb{Q}) = 0$ and there exist at most countably linearly independent elements $\{\gamma_i\}_{2\leqslant i<N}$ in $H^{\text{even}}_c(M,\mathbb{Q})$ such that $H^{\text{even}}_c(M,\mathbb{Q}) = \text{span}(\{\gamma_i\}_{2\leqslant i<N})$. Here N is a natural number or $+\infty$. Set $\gamma_1 = \mathbf{1}$. For $1 \leqslant a, b < N$ let

$$\zeta_{ab} = \int_M \gamma_a^* \wedge \gamma_b^* \quad \text{if } \deg\gamma_a + \deg\gamma_b = 2n, \quad \text{and} \quad \zeta_{ab} = 0 \quad \text{otherwise.} \tag{3}$$

With these $\{\gamma_i\}_{1\leqslant i<N}$ and formal variables t^a_r, $1 \leqslant a < N$, $r = 0,1,2,\ldots,$ all genus-g correlators can be assembled into a generating function, called *free energy function* [5], as follows:

$$F^M_g\big(t^a_r; q\big) = \sum_{n_{r,a}} \prod_{r,a} \frac{(t^a_r)^{n_{r,a}}}{n_{r,a}!} \bigg\langle \prod_{r,a} \tau^{n_{r,a}}_{r,\gamma_a} \bigg\rangle_g (q), \tag{4}$$

where $n_{r,a}$ are arbitrary collections of nonnegative integers, almost all zero, labelled by r, a. Witten's generating function [5] is the infinite sum

$$F^M\big(t^a_r; q\big) = \sum_{g\geqslant 0} \lambda^{2g-2} F^M_g\big(t^a_r; q\big), \tag{5}$$

where λ is the genus expansion parameter. As in [5,4] we can derive that the functions $F^M(t^a_r; q)$ and $F^M_g(t^a_r; q)$ satisfy respectively the generalized string equation and the dilation equation (replacing η_{ab} in Lemma 6.1 of [4] by ζ_{ab} in (3)). More generally, for a given collection of nonzero homogeneous elements $\underline{\xi} = \{\xi_i\}_{1\leqslant i\leqslant l}$ in $H^*_c(M,\mathbb{C}) \cup H^*(M,\mathbb{C})$ we replace (2) by

$$\langle \underline{\xi} | \tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_m,\alpha_m}\rangle_{g}(q) = \sum_{A\in H_2(M)} \mathcal{GW}^{(\omega,\mu,J)}_{A,g,m+l}(\kappa_{d_1,\ldots,d_m}; \alpha_1,\ldots,\alpha_m, \underline{\xi}) q^A \tag{6}$$

and make the following

Convention 1. $\mathcal{GW}^{(\omega,\mu,J)}_{A,g,m+2}(\kappa; \mathbf{1},\mathbf{1},\alpha_1,\ldots,\alpha_m) = 0$ for any $m \geqslant 0$ whether or not this case may be included in our definition category. (This is reasonable if it may be defined.)

Theorem 1.1. *The variants of* (4), (5),

$$F^M_g\big(\underline{\xi}|t^a_r; q\big) = \sum_{n_{r,a}} \prod_{r,a} \frac{(t^a_r)^{n_{r,a}}}{n_{r,a}!} \bigg\langle \underline{\xi} \bigg| \prod_{r,a} \tau^{n_{r,a}}_{r,\gamma_a} \bigg\rangle_g (q), \qquad F^M\big(\underline{\xi}|t^a_r; q\big) = \sum_{g\geqslant 0} \lambda^{2g-2} F^M_g\big(\underline{\xi}|t^a_r; q\big),$$

still called Witten's generating function, *satisfy, respectively*

$$\frac{\partial F^M(\underline{\xi}|\cdot)}{\partial t_0^1}=\sum_{i=0}^{\infty}\sum_a t_{i+1}^a\frac{\partial F^M(\underline{\xi}|\cdot)}{\partial t_i^a},\qquad \frac{\partial F_g^M(\underline{\xi}|\cdot)}{\partial t_1^1}=\left(2g-2+\sum_{i=1}^{\infty}\sum_a t_i^a\frac{\partial}{\partial t_i^a}\right)F_g^M(\underline{\xi}|\cdot).$$

These are still called the generalized string equation and dilation equation. *Moreover, if* $c_1(M)=0$*,* $F^M(\underline{\xi}|\cdot)$ *also satisfies the dilation equation*

$$\frac{\partial F^M(\underline{\xi}|\cdot)}{\partial t_1^1}=\sum_{i=1}^{\infty}\sum_a\left(\frac{2}{3-n}\left(i-1+\frac{1}{2}\deg\gamma_a\right)+1\right)t_i^a\frac{\partial F^M(\underline{\xi}|\cdot)}{\partial t_i^a}.$$

2. WDVV equation and quantum cohomology

It is well known that the quantum cohomology ring of a closed symplectic manifold provides an example of Witten's topological σ-model [5]. However, for noncompact geometrically bounded symplectic manifolds we need to assume

$$\dim H^*(M)<\infty \tag{7}$$

so that our Gromov–Witten invariant theory in the previous note [1] can be used to get the desired WDVV equation and quantum products on cohomology groups.

Let $\{\beta_i\}_{1\leqslant i\leqslant L}$ be a basis of $H^*(M,\mathbb{Q})$ consisting of homogeneous elements as in Theorem 2.2 in [1] (or Lemma 5.4 in [2]). We may assume that $\deg\beta_i$ is even if and only if $i\leqslant P$. Let $\underline{\alpha}=\{\alpha_i\}_{1\leqslant i\leqslant k}$ be a collection of nonzero homogeneous elements in $H_c^*(M,\mathbb{C})\cup H^*(M,\mathbb{C})$, at least one of them belonging to $H_c^*(M,\mathbb{C})$. Putting $w=\sum_{i=1}^P t_i\beta_i\in W=H^{\mathrm{even}}(M,\mathbb{C})$ we define $\underline{\alpha}$*-Gromov–Witten potential* by a formal power series in (a specified number) q,

$$\Phi_{(q,\underline{\alpha})}(w)=\sum_{A\in H_2(M)}\sum_{m\geqslant\max(1,3-k)}\frac{1}{m!}\mathcal{GW}_{A,0,k+m}^{(\omega,\mu,J)}\big([\overline{\mathcal{M}}_{0,k+m}];\underline{\alpha},w,\ldots,w\big)q^A. \tag{8}$$

Theorem 2.1. *The function* $\Phi_{(q,\underline{\alpha})}$ *satisfies* WDVV*-equation of the following form*

$$\sum_{r,s}\frac{\partial^3\Phi_{(q,\underline{\alpha})}}{\partial t_i\partial t_j\partial t_r}\eta^{rs}\frac{\partial^3\Phi_{(q,\underline{\alpha})}}{\partial t_k\partial t_l\partial t_s}=\sum_{r,s}\frac{\partial^3\Phi_{(q,\underline{\alpha})}}{\partial t_i\partial t_k\partial t_r}\eta^{rs}\frac{\partial^3\Phi_{(q,\underline{\alpha})}}{\partial t_j\partial t_l\partial t_s} \tag{9}$$

for $1\leqslant i,j,k,l\leqslant P$*, where* $\eta^{rs}=\int_M\omega_r\wedge\omega_s$ *as in Theorem* 2.2 *in* [1] (*or Lemma* 5.4 *in* [2]).

As in [3] we can use (9) to get a family of the flat connections $\{\nabla^\epsilon\}$ on the tangent bundle TW. Under assumption (7) we may also define the quantum cohomology. Let $A_1,\ldots,A_d$ be a finite integral basis of the free part $H_2(M)$ of $H_2(M,\mathbb{Z})$, and $q_j=\mathrm{e}^{2\pi iA_j}$, $j=1,\ldots,d$. For every $A=r_1A_1+\cdots+r_dA_d\in H_2(M)$ we denote by $q^A=q_1^{r_1}\cdots q_d^{r_d}$. As usual we have the Novikov ring $\Lambda_\omega(\mathbb{Q})$ over $\mathbb{Q}$ and $QH^*(M,\mathbb{Q}):=H^*(M,\mathbb{Q})\otimes\Lambda_\omega(\mathbb{Q})$. Let $\{\beta_i\}_{1\leqslant i\leqslant L}$ and $\underline{\alpha}$ be as in (8). For $\alpha,\beta\in H^*(M,\mathbb{Q})$ we define an element of $QH^*(M,\mathbb{Q})$ by

$$\alpha\star_{\underline{\alpha}}\beta=\sum_{A\in H_2(M)}\sum_{i,j}\mathcal{GW}_{A,0,3+k}^{(\omega,\mu,J)}\big([\overline{\mathcal{M}}_{0,3+k}];\underline{\alpha},\alpha,\beta,\beta_i\big)\eta^{ij}\beta_jq^A. \tag{10}$$

More generally, for a given $w=\sum_{i=1}^L t_i\beta_i\in H^*(M,\mathbb{C})$ we also define another element of $QH^*(M,\mathbb{C})=H^*(M,\mathbb{C})\otimes\Lambda_\omega(\mathbb{C})$ by

$$\alpha \star_{(\underline{\alpha},w)} \beta = \sum_{A\in H_2(M)} \sum_{k,l} \sum_{m\geqslant 0} \frac{\epsilon(\{t_i\})}{m!} \times \mathcal{GW}^{(\omega,\mu,J)}_{A,0,3+k+m}\big([\overline{\mathcal{M}}_{0,3+k+m}]; \underline{\alpha}, \alpha, \beta, \beta_k, \beta_{i_1}, \ldots, \beta_{i_m}\big)\eta^{kl}\beta_l t_{i_1}\cdots t_{i_m} q^A, \tag{11}$$

where $\epsilon(\{t_i\})$ is the sign of the induced permutation on odd dimensional β_i. Clearly, (10) is the special case of (11) at $w=0$. We still call the operations defined by (10) and (11) the 'small quantum product' and the 'big quantum product', respectively. However, it is unpleasant that both $\alpha \star_{\underline{\alpha}} \mathbf{1}$ and $\alpha \star_{(\underline{\alpha},w)} \mathbf{1}$ are always zero by Theorem 4.1 in [2]. After extending them to $QH^*(M,\mathbb{C}) = H^*(M,\mathbb{C}) \otimes \Lambda_\omega(\mathbb{C})$ by linearity over $\Lambda_\omega(\mathbb{C})$ we can derive from Theorem 2.2 in [1] that

$$(\alpha \star_{(\underline{\alpha},w)} \beta) \star_{(\underline{\alpha},w)} \gamma = \alpha \star_{(\underline{\alpha},w)} (\beta \star_{(\underline{\alpha},w)} \gamma)$$

for any $\alpha, \beta, \gamma \in H^*(M,\mathbb{C})$. Consequently, $QH^*(M,\mathbb{C})$ is a supercommutative ring without identity under the quantum products in (10) and (11).

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