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Partial Differential Equations

Extremal singular solutions for degenerate logistic-type equations in anisotropic media

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Abstract

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Let $b \geqslant 0$, $b \not\equiv 0$ be a continuous function on $\overline{\Omega}$ and consider a closed subset $D_0 \neq \emptyset$ of $[b = 0]$. We study the logistic problem $\Delta u + au = b(x)f(u)$ in $\Omega \setminus D_0$, $\mathcal{B}u = 0$ on $\partial\Omega$, and $u = +\infty$ on ∂D_0, where a is a real number, $\mathcal{B}$ denotes either the Dirichlet or the mixed boundary operator, and $f \geqslant 0$ is a smooth function such that $f(u)/u$ is increasing on $(0, \infty)$. In this Note we establish the existence of extremal singular solutions to the above problem, a uniqueness result, and we describe the blow-up at the boundary. ***To cite this article: F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Solutions singulières extremales des équations du type logistique en milieu anisotrope. Soit Ω un domaine borné et régulier de $\mathbb{R}^N$. Soit $b \geqslant 0$, $b \not\equiv 0$ une fonction continue dans $\overline{\Omega}$ et $D_0 \neq \emptyset$ un sous-ensemble fermé de $[b = 0]$. On étudie le problème logistique $\Delta u + au = b(x)f(u)$ dans $\Omega \setminus D_0$, $\mathcal{B}u = 0$ sur $\partial\Omega$, et $u = +\infty$ sur ∂D_0, où a est un réel, $\mathcal{B}$ désigne ou bien une condition de Dirichlet ou bien une condition mixte sur $\partial\Omega$, et $f \geqslant 0$ est une fonction régulière telle que l'application $f(u)/u$ soit croissante sur $(0, \infty)$. Dans cette Note on établit l'existence des solutions singulières extremales, un résultat d'unicité et on décrit également la vitesse d'explosion au bord. ***Pour citer cet article : F.-C. Cîrstea, V. Rădulescu, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Soit Ω un domaine borné et régulier de $\mathbb{R}^N$, $N \geqslant 3$. On désigne par $\mathcal{B}$ l'opérateur de Dirichlet $\mathcal{D}u := u$ ou bien l'opérateur de Neumann/Robin $\mathcal{R}u = u_\nu + \beta(x)u$ sur $\partial\Omega$, où ν est le vecteur unité de la normale extérieure sur $\partial\Omega$ et $0 \leqslant \beta \in C^{1,\mu}(\partial\Omega)$. Soit $b \in C^{0,\mu}(\overline{\Omega})$ $(0 < \mu < 1)$ une fonction non négative dans Ω telle que $b > 0$ sur $\partial\Omega$ si $\mathcal{B} = \mathcal{R}$. On définit $\Omega_{0,b} := \{x \in \Omega\colon\ b(x) = 0\}$ et on suppose que $\Omega_{0,b} = D_0 \cup \overline{D}_1$, où $D_0 \neq \emptyset$ est un ensemble fermé tel que $\Omega \setminus D_0$ soit connexe et $D_1 \subset\subset \Omega \setminus D_0$ est un ensemble connexe. On suppose que ∂D_1 est régulier (éventuellement vide). Soit $\lambda_{\infty,1}(D_1)$ la première valeur propre de $(-\Delta)$ dans $H_0^1(D_1)$, avec la convention $\lambda_{\infty,1}(D_1) = +\infty$ si $D_1 = \emptyset$. On considère le problème elliptique singulier

$$\Delta u + au = b(x)f(u) \quad \text{dans } \Omega \setminus D_0, \qquad \mathcal{B}u = 0 \quad \text{sur } \partial\Omega, \qquad u = +\infty \quad \text{sur } \partial D_0, \tag{P}$$

où $a \in \mathbb{R}$, $f \in C^1[0,\infty)$, $f \geqslant 0$ et l'application $f(u)/u$ est strictement croissante sur $(0,\infty)$.

On démontre d'abord

Théorème 0.1. *Supposons, de plus, que f satisfait la condition de Keller–Osserman $\int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$, où $F(t) = \int_0^t f(s)\,\mathrm{d}s$. Si le problème* (P) *a une solution non négative, alors $a < \lambda_{\infty,1}(D_1)$ et, dans ce cas, le problème admet une solution minimale et une solution maximale qui sont positives dans Ω.*

Soit $\mathcal{K}$ l'ensemble des fonctions $k:(0,\nu) \to (0,\infty)$ (pour un certain ν), de classe C^1, croissantes, telles que $\lim_{t\searrow 0}(\int_0^t k(s)\,\mathrm{d}s/k(t))^{(i)} := \ell_i$, pour $i = \overline{0,1}$. On définit $\Lambda(u) := u\int_0^{1/u} k(s)\,\mathrm{d}s/k(1/u)$, où $u > 1/\nu$. Soit $\mathbb{R}_q$ la classe des fonctions à variation régulière à l'infini d'indice $q \in \mathbb{R}$ (voir [2]). Pour la notion de Γ-variation à l'infini voir [6]. On écrit $d(x) := \operatorname{dist}(x, D_0)$. On démontre le résultat suivant d'unicité.

Théorème 0.2. *Supposons que $f' \in \mathbb{R}_\rho$ $(\rho > 0)$ et que $\lim_{d(x)\searrow 0} b(x)/k^2(d(x)) = c$, pour $c > 0$ et $k \in \mathcal{K}$. Alors, pour chaque $a < \lambda_{\infty,1}(D_1)$, le problème* (P) *admet une seule solution positive u_a et, de plus, $\lim_{d(x)\searrow 0} \frac{u_a(x)}{h(d(x))} = \xi_0$, où $\xi_0 = (\frac{2+\ell_1\rho}{c(2+\rho)})^{1/\rho}$ et $\int_{h(t)}^\infty \frac{\mathrm{d}s}{\sqrt{2F(s)}} = \int_0^t k(s)\,\mathrm{d}s$, $\forall t \in (0,\nu)$.*

Si $\ell_1 \neq 0$, alors $h(1/u) \in \mathbb{R}_{2/(\rho\ell_1)}$, i.e., il existe $L \in \mathbb{R}_0$ tel que $\lim_{d(x)\searrow 0} u_a(x)[d(x)]^{2/(\rho\ell_1)} L(1/d(x)) = 1$.

Si $\ell_1 = 0$, alors $h(1/u)$ a une Γ-variation à l'infini avec la fonction auxiliaire $g(u) = \rho u\Lambda(u)/2$. Si, de plus, $\Lambda(u) \in \mathbb{R}_j$ $(j \leqslant 0)$, alors il existe $T \in \mathbb{R}_{-2/\rho}$ et $W \in \mathbb{R}_{-j}$ tels que $\lim_{d(x)\searrow 0} u_a(x)T(\mathrm{e}^{W(1/d(x))}) = 1$.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ be a smooth bounded domain. Denote by $\mathcal{B}$ either the Dirichlet boundary operator $\mathcal{D}u := u$ or the Neumann/Robin boundary operator $\mathcal{R}u = u_\nu + \beta(x)u$ where ν is the unit outward normal to $\partial\Omega$ and $\beta \geqslant 0$ is in $C^{1,\mu}(\partial\Omega)$ with $0 < \mu < 1$. Let $a \in \mathbb{R}$ and $b \in C^{0,\mu}(\overline{\Omega})$ satisfy $b \geqslant 0$, $b \not\equiv 0$ in Ω.

Set $\Omega_{0,b} := \{x \in \Omega :\ b(x) = 0\}$. We assume that $\Omega_{0,b} = D_0 \cup \overline{D}_1$, where $D_0 \neq \emptyset$ is a closed set such that $\Omega \setminus D_0$ is connected with smooth boundary, and $D_1 \subset\subset \Omega \setminus D_0$ is a connected set.

We are concerned with the existence and uniqueness for the singular mixed boundary blow-up problem

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega \setminus D_0, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \qquad u = \infty \quad \text{on } \partial D_0 \tag{1}$$

where $u = \infty$ on ∂D_0 means that $u(x) \to \infty$ as $x \in \Omega \setminus D_0$ and $d(x) := \operatorname{dist}(x, D_0) \to 0$.

Suppose $b > 0$ on $\partial\Omega$ if $\mathcal{B} = \mathcal{R}$ and ∂D_1 satisfies the exterior cone condition (possibly, $D_1 = \emptyset$). Let $\lambda_{\infty,1}(D_1)$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(D_1)$. Set $\lambda_{\infty,1}(D_1) = \infty$ if $D_1 = \emptyset$.

For the significance of (1) in the case $f(u) = u^p$ $(p > 1)$ and $\Omega_{0,b} = D_0$ we refer to Du and Huang [4].

Our aim is to improve the existence and uniqueness results of (1) which are communicated in [1,2,4].

By (A_1) we mean that $0 \leqslant f \in C^1[0,\infty)$ and $f(u)/u$ is increasing on $(0,\infty)$. By [1, Remark 3.1], a positive blow-up boundary solution of $\Delta u + au = b(x)f(u)$ in Ω may exist only if the Keller–Osserman condition (in short (A_2)) is fulfilled i.e., $\int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$, $t > 0$.

Theorem 1.1. *Let* (A_1) *and* (A_2) *hold. If* (1) *has a nonnegative solution, then* $a < \lambda_{\infty,1}(D_1)$*. Furthermore, for any* $a < \lambda_{\infty,1}(D_1)$*, there exists a minimal* (*resp., maximal*) *positive solution of* (1).

Let $\mathbb{R}_q$ denote the set of all functions that are regularly varying at infinity with index $q \in \mathbb{R}$ (see [2]).

Definition 1.2 (see [6]). A nondecreasing function U is Γ-varying if U is defined on an interval (x_l, x_0), $\lim_{x \nearrow x_0} U(x) = \infty$ and there is $g : (x_l, x_0) \to (0, \infty)$ such that $\lim_{y \to x_0} U(y + \lambda g(y))/U(y) = e^{\lambda}$, $\forall \lambda \in \mathbb{R}$.

The function g is called an *auxiliary function* and is unique up to asymptotic equivalence. For the basic definitions and main properties of regularly (resp., Γ)-varying functions we refer to [6,7].

Let $\mathcal{K}$ be the set of all positive, nondecreasing $k \in C^1(0,\nu)$ satisfying $\lim_{t \searrow 0} (\int_0^t k(s)\,ds / k(t))^{(i)} := \ell_i$, with $i = \overline{0,1}$. Note that, for every $k \in \mathcal{K}$, $\ell_0 = 0$ and $\ell_1 \in [0,1]$. A complete characterisation of $\mathcal{K}$ is provided by [3]. Recall that $k \in \mathcal{K}$ with $\ell_1 = 0$ if and only if $k \in \mathcal{R}_0$ where

$$\mathcal{R}_0 = \left\{ \begin{array}{l} k\colon\ k(1/u) = d_0 u \big[\Lambda(u)\big]^{-1} e^{-\int_{d_1}^{u} [s\Lambda(s)]^{-1}\,ds}\ (u \geqslant d_1),\ \text{where } 0 < \Lambda \in C^1[d_1,\infty), \\ \lim_{u\to\infty} \Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0 \text{ and } d_i > 0 \text{ are constants} \end{array} \right\}.$$

Remark 1. We have $\{k \in \mathcal{R}_0\colon\ \Lambda \in \mathbb{R}_j\ (j < 0)\} \equiv \{k \in \mathcal{K}\colon\ \lim_{t\to 0} \frac{tk'(t)}{k(t)\ln k(t)} = j < 0\} \equiv \mathcal{M}$, where $\mathcal{M} := \{k\colon\ k(1/u) = e^{-S(u)}\ (u \geqslant D > 0)$ for some $S \in C^1[D,\infty),\ S' \in \mathbb{R}_q$ with $q > -1\}$.

Theorem 1.3. *Let* (A_1) *hold and* $f' \in \mathbb{R}_\rho$ $(\rho > 0)$*. Suppose* $\lim_{d(x)\searrow 0} \frac{b(x)}{k^2(d(x))} = c$ *for some constant* $c > 0$ *and* $k \in \mathcal{K}$*. Then, for any* $a < \lambda_{\infty,1}(D_1)$*,* (1) *has a unique positive solution* u_a*. Moreover,*

$$\lim_{d(x)\searrow 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \quad \text{where } \xi_0 = \left(\frac{2+\ell_1\rho}{c(2+\rho)}\right)^{1/\rho} \text{ and } \int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds,\ \forall t \in (0,\nu). \tag{2}$$

If $\ell_1 \neq 0$*, then* $h(1/u) \in \mathbb{R}_{2/(\rho\ell_1)}$*, i.e., there exists* $L(u) \in \mathbb{R}_0$*, such that*

$$\lim_{d(x)\searrow 0} u_a(x)\big[d(x)\big]^{2/(\rho\ell_1)} L\big(1/d(x)\big) = 1, \quad \forall a < \lambda_{\infty,1}(D_1). \tag{3}$$

If $\ell_1 = 0$*, then* $h(1/u)$ *is* Γ*-varying at* $u = \infty$ *with auxiliary function* $g(u) = \rho u \Lambda(u)/2$*. When* $\Lambda \in \mathbb{R}_j$ $(j \leqslant 0)$*, then there exists* $T \in \mathbb{R}_{-2/\rho}$ *and* $W \in \mathbb{R}_{-j}$ *such that*

$$\lim_{d(x)\searrow 0} u_a(x) T\big(e^{W(1/d(x))}\big) = 1, \quad \forall a < \lambda_{\infty,1}(D_1). \tag{4}$$

The novelties brought by our Note are the following:

(a) We allow b to vanish on $\Omega \setminus D_0$. Moreover, in the case $\mathcal{B} = \mathcal{D}$, we remove the assumption $b > 0$ on $\partial\Omega$ which is made in [1,4]. Theorem 1.1 shows that the existence of positive solutions to (1) takes place if and only if the parameter a is suitably connected with the zero set of b in $\Omega \setminus D_0$.

(b) Theorem 1.1 improves [1, Theorem 1.2] (resp., [4, Theorem 2.4]) where $b > 0$ on $\overline{\Omega} \setminus D_0$ and the additional hypothesis $\lim_{u\to\infty} (F/f)'(u) = \gamma$ (resp., $f(u) = u^p$, $p > 1$) was required. By treating the degenerate case for b, Lemmas 2.1 and 2.2 extend the comparison principles (Lemmas 2.1 and 2.3) in [1].

(c) Theorem 1.3 and [2, Lemma 1] show that the claim of [1, Theorem 1.3] follows without requiring (A_3) and (A_4). Note that the condition $\lim_{d(x)\searrow 0} \frac{b(x)}{[d(x)]^\alpha} = c_1$ for some constants $c_1, \alpha > 0$ (imposed in [4, Theorem 2.8]) is *not* necessary for the uniqueness (use, for instance, Theorem 1.3 with $k \in \mathcal{M}$).

(d) Relations (3) and (4) offer a new insight into the asymptotic behaviour of u_a near $\partial\Omega$. This relies on [3, Proposition 2] and some properties of regularly (resp., Γ)-varying functions in [6].

2. Proofs

In Lemmas 2.1 and 2.2 we assume that f is continuous on $(0,\infty)$ and $f(u)/u$ is increasing for $u>0$.

Lemma 2.1. *Let $D\subset\mathbb{R}^N$ be a bounded domain and $0\not\equiv p\in C(D)$ be a nonnegative function. If $u_1,u_2\in C^2(D)$ are positive such that $\limsup_{\operatorname{dist}(x,\partial D)\to 0}(u_2-u_1)(x)\leqslant 0$ and*

$$\Delta u_1+au_1-p(x)f(u_1)\leqslant 0\leqslant \Delta u_2+au_2-p(x)f(u_2)\quad \text{in } D, \tag{5}$$

then $u_1\geqslant u_2$ on D.

Proof. We use here some ideas and approximation techniques introduced by Marcus and Véron in [5, Lemma 1.1]. Set $\mathcal{O}=\{x\in D\colon\ u_1(x)<u_2(x)\}$. Of course, $u_1\geqslant u_2$ on D is equivalent to $\mathcal{O}=\emptyset$.

Let ϕ_1,ϕ_2 be two nonnegative C^2-functions on $\overline{D}$ vanishing near ∂D. Using (5) we have

$$\int_D(\nabla u_2\cdot\nabla\phi_2-\nabla u_1\cdot\nabla\phi_1)\,\mathrm{d}x+\int_D p(x)\big(f(u_2)\phi_2-f(u_1)\phi_1\big)\,\mathrm{d}x\leqslant a\int_D(u_2\phi_2-u_1\phi_1)\,\mathrm{d}x. \tag{6}$$

Fix $\varepsilon>0$. Set $D_\varepsilon=\{x\in D\colon\ u_2(x)>u_1(x)+\varepsilon\}$ and $v_i=(u_i+2\varepsilon/i)^{-1}((u_2+\varepsilon)^2-(u_1+2\varepsilon)^2)^+$ for $i=1,2$. We see that $v_i\in H^1(D)$ and it vanishes outside the set D_ε. Since $\limsup_{\operatorname{dist}(x,\partial D)\to 0}(u_2-u_1)(x)\leqslant 0$, we have $D_\varepsilon\subset\subset D$. Hence, v_i can be approximated closely in the $H^1\cap L^\infty$ topology on $\overline{D}$ by nonnegative C^2 functions vanishing near ∂D. It follows that Eq. (6) holds with v_i instead of ϕ_i. Precisely, (6) becomes

$$\int_{D_\varepsilon}(\nabla u_2\cdot\nabla v_2-\nabla u_1\cdot\nabla v_1)\,\mathrm{d}x+\int_{D_\varepsilon} p(x)\big(f(u_2)v_2-f(u_1)v_1\big)\,\mathrm{d}x\leqslant a\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x. \tag{7}$$

Let $\tau\in(0,1)$ be arbitrary. For any $\varepsilon\in(0,\tau)$, we have

$$0\leqslant\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x=\int_{D_\tau}(u_2v_2-u_1v_1)\,\mathrm{d}x+\int_{D_\varepsilon\setminus D_\tau}(u_2v_2-u_1v_1)\,\mathrm{d}x. \tag{8}$$

But $\overline{D}_\tau\subset D$ yields $\max_{\overline{D}_\tau}u_2=M_d<\infty$ and $\min_{\overline{D}_\tau}u_1=m_d>0$. Thus, for any $x\in D_\tau$, we obtain $0<u_2/(u_2+\varepsilon)-u_1/(u_1+2\varepsilon)\leqslant 1-m_d/(m_d+2\varepsilon)=2\varepsilon/(m_d+2\varepsilon)\to 0$ as $\varepsilon\to 0$. Consequently, $u_2/(u_2+\varepsilon)-u_1/(u_1+2\varepsilon)\to 0$ as $\varepsilon\to 0$ uniformly on D_τ. It follows that

$$0\leqslant\int_{D_\tau}(u_2v_2-u_1v_1)\,\mathrm{d}x\leqslant(M_d+1)^2\int_{D_\tau}\left(\frac{u_2}{u_2+\varepsilon}-\frac{u_1}{u_1+2\varepsilon}\right)\mathrm{d}x\to 0\quad\text{as }\varepsilon\to 0. \tag{9}$$

We see that $u_2\in(u_1+\varepsilon,u_1+\tau]$ on $D_\varepsilon\setminus D_\tau$. Thus, for each $x\in D_\varepsilon\setminus D_\tau$, we have $0<u_2v_2-u_1v_1=[2\varepsilon/(u_1+2\varepsilon)-\varepsilon/(u_2+\varepsilon)][(u_2+\varepsilon)^2-(u_1+2\varepsilon)^2]\leqslant[2(u_1+\varepsilon)(\tau-\varepsilon)+\tau^2-\varepsilon^2]2\varepsilon/(u_1+2\varepsilon)\leqslant 2\varepsilon[2(\tau-\varepsilon)+(\tau^2-\varepsilon^2)/(2\varepsilon)]<5\tau^2$. Hence, $\limsup_{\varepsilon\to 0}\int_{D_\varepsilon\setminus D_\tau}(u_2v_2-u_1v_1)\,\mathrm{d}x\leqslant 5\tau^2|D|$. By (8) and (9), $0\leqslant\liminf_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x\leqslant\limsup_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x\leqslant 5\tau^2|D|$. It follows that $\lim_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x=0$.

Assume by contradiction that $\mathcal{O}\neq\emptyset$. For $x_0\in\mathcal{O}$ arbitrary, there exists a small closed ball $B=B(x_0)$ centred at x_0 such that $B\subset\mathcal{O}$. Since $\min_B(u_2-u_1)=m_B>0$, we get $B\subset D_\varepsilon$, $\forall\varepsilon\in(0,m_B)$. It is easy to check that $\nabla u_2\nabla v_2-\nabla u_1\nabla v_1=|(u_2+\varepsilon)^{-1}\nabla u_2-(u_1+2\varepsilon)^{-1}\nabla u_1|^2[(u_2+\varepsilon)^2+(u_1+2\varepsilon)^2]\geqslant 0$ on D_ε.

Since $f(t)/(t+\varepsilon)$ is increasing on $(0,\infty)$, we find $f(u_1)/(u_1+2\varepsilon)<f(u_1+\varepsilon)/(u_1+2\varepsilon)<f(u_2)/(u_2+\varepsilon)$ on D_ε. Thus, all the integrands in the left-hand side of (7) are nonnegative. So, for each $\varepsilon\in(0,m_B)$, we have $0\leqslant\int_B(\nabla u_2\cdot\nabla v_2-\nabla u_1\cdot\nabla v_1)\,\mathrm{d}x+\int_B p(x)(f(u_2)v_2-f(u_1)v_1)\,\mathrm{d}x\leqslant a\int_{D_\varepsilon}(u_2v_2-u_1v_1)\,\mathrm{d}x$. Letting $\varepsilon\searrow 0$, we get $\frac{\nabla u_2(x)}{u_2(x)}=\frac{\nabla u_1(x)}{u_1(x)}$ and $p(x)=0$, for each $x\in B\ni x_0$. Since $x_0\in\mathcal{O}$ is arbitrary, we find $\nabla(\ln u_2-\ln u_1)=0$ and $p\equiv 0$ on $\mathcal{O}$. But $p\not\equiv 0$ in D so that $\mathcal{O}\neq D$. Thus, $\partial\mathcal{O}\cap D\neq\emptyset$. Let $z\in\partial\mathcal{O}\cap D$ and $\mathcal{C}$ be a domain included in $\mathcal{O}$ so that $z\in\partial\mathcal{C}$. Hence $u_1(z)=u_2(z)$ and $\nabla(\ln u_2-\ln u_1)\equiv 0$ on $\mathcal{C}$, i.e., $u_2/u_1=\text{Const.}>0$ on $\mathcal{C}$. By the continuity of u_i, we obtain $u_1=u_2$ on $\mathcal{C}$. This contradicts $\mathcal{C}\subseteq\mathcal{O}$. □

Lemma 2.2. *Let $\omega \subset\subset \Omega$ and $0 \not\equiv p \in C(\overline{\Omega} \setminus \omega)$ be a nonnegative function.*

If $u_1, u_2 \in C^2(\overline{\Omega} \setminus \overline{\omega})$ are positive functions in $\Omega \setminus \overline{\omega}$ such that $\limsup_{\mathrm{dist}(x,\partial\omega)\to 0}(u_2 - u_1)(x) \leqslant 0$ *and*

$$\Delta u_1 + au_1 - p(x)f(u_1) \leqslant 0 \leqslant \Delta u_2 + au_2 - p(x)f(u_2) \quad \text{in } \Omega \setminus \overline{\omega} \tag{10}$$

$$\text{either } \mathcal{B}u_1 \geqslant \mathcal{B}u_2 \quad \text{on } \partial\Omega \text{ if } \mathcal{B} = \mathcal{D} \quad \text{or} \quad \mathcal{B}u_1 \geqslant 0 \geqslant \mathcal{B}u_2 \quad \text{on } \partial\Omega \text{ if } \mathcal{B} = \mathcal{R} \tag{11}$$

then $u_1 \geqslant u_2$ on $\overline{\Omega} \setminus \overline{\omega}$.

Proof. If $\mathcal{B} = \mathcal{D}$, then the assertion follows by Lemma 2.1. Suppose $\mathcal{B} = \mathcal{R}$. Set $D := \Omega \setminus \overline{\omega}$ and define $\mathcal{O}$ as in Lemma 2.1. Assume by contradiction that $\mathcal{O} = \emptyset$.

Let ϕ_1, ϕ_2 be two nonnegative C^2-functions on $\overline{\Omega} \setminus \omega$ vanishing near $\partial\omega$. Using (10) and (11), we find $\int_D (\nabla u_2 \nabla\phi_2 - \nabla u_1 \nabla\phi_1) + p(f(u_2)\phi_2 - f(u_1)\phi_1)\,\mathrm{d}x + \int_{\partial\Omega} \beta(u_2\phi_2 - u_1\phi_1)\,\mathrm{d}S \leqslant a\int_D (u_2\phi_2 - u_1\phi_1)\,\mathrm{d}x$. Let D_ε and v_i be as in the proof of Lemma 2.1. The above relation holds with D, ϕ_1 and ϕ_2 respectively, replaced by D_ε, v_1 and v_2 respectively. For $\tau \in (0,1)$ arbitrary, set $G_\tau = \{x \in D_\tau\colon \mathrm{dist}(x,\partial\Omega) \geqslant \tau\}$, $L_\tau = \{x \in \Omega\colon \mathrm{dist}(x,\partial\Omega) < \tau\}$ and $K_{\varepsilon\tau} = \{x \in D_\varepsilon\colon \mathrm{dist}(x,\partial\Omega) < \tau\}$. For any $\varepsilon \in (0,\tau)$, we have

$$0 \leqslant \int_{D_\varepsilon} (u_2v_2 - u_1v_1)\,\mathrm{d}x \leqslant \int_{K_{\varepsilon\tau}} (u_2v_2 - u_1v_1)\,\mathrm{d}x + \int_{G_\tau} (u_2v_2 - u_1v_1)\,\mathrm{d}x + \int_{D_\varepsilon\setminus D_\tau} (u_2v_2 - u_1v_1)\,\mathrm{d}x. \tag{12}$$

As in Lemma 2.1, $u_2/(u_2+\varepsilon) - u_1/(u_1+2\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly on G_τ. We also deduce $\lim_{\varepsilon\to 0}\int_{G_\tau}(u_2v_2 - u_1v_1)\,\mathrm{d}x = 0$ (see (9)) and $\limsup_{\varepsilon\to 0}\int_{D_\varepsilon\setminus D_\tau}(u_2v_2 - u_1v_1)\,\mathrm{d}x \leqslant 5\tau^2|D|$. Note that $\int_{K_{\varepsilon\tau}}(u_2v_2 - u_1v_1)\,\mathrm{d}x \leqslant 2\max_{x\in\overline{L}_\tau}(u_2(x)+1)^2|L_\tau|$. By (12), we find $0 \leqslant \liminf_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2 - u_1v_1)\,\mathrm{d}x \leqslant \limsup_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2 - u_1v_1)\,\mathrm{d}x \leqslant 2\max_{x\in\overline{L}_\tau}(u_2+1)^2|L_\tau| + 5\tau^2|D|$. Since $|D| < \infty$ and $|L_\tau| \to 0$ as $\tau \to 0$, we regain $\lim_{\varepsilon\to 0}\int_{D_\varepsilon}(u_2v_2 - u_1v_1)\,\mathrm{d}x = 0$. The same argument used before leads to a contradiction. □

Lemma 2.3. *Assume* (A_1) *and* (A_2) *hold. If $0 \not\equiv \Phi \in C^{2,\mu}(\partial D_0)$ is a nonnegative function, then*

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega \setminus D_0, \qquad \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \qquad u = \Phi \quad \text{on } \partial D_0 \tag{13}$$

has a positive solution if and only if $a \in (-\infty, \lambda_{\infty,1}(D_1))$; in this case, the solution is unique.

Proof. Let $\widetilde{\Omega}$ be a smooth subdomain of $\Omega \setminus D_0$ so that $b|_{\partial\widetilde{\Omega}} > 0$ and $\overline{D}_1 \subset \widetilde{\Omega}$. If $u_\mathcal{B}$ is a positive solution of (13), then it satisfies $\Delta u + au = b(x)f(u)$ in $\widetilde{\Omega}$, $u|_{\partial\widetilde{\Omega}} = u_\mathcal{B}$. By [1, Lemma 3.2], we get $a < \lambda_{\infty,1}(D_1)$.

Fix $a < \lambda_{\infty,1}(D_1)$. Let v_∞ be a positive blow-up boundary solution of $\Delta u + au = b(x)f(u)$ in $\Omega \setminus D_0$ (see [1, Theorem 1.1]). Let $\delta > 0$ be small such that $b > 0$ on $T_{2\delta} := \{x \in \Omega\colon \mathrm{dist}(x,\partial\Omega) < 2\delta\}$. Set $C_\delta = \{y \in \mathbb{R}^N\colon \mathrm{dist}(y,\partial\Omega) < \delta\}$. Let $p \in C^{0,\mu}(\overline{C}_\delta)$ be such that $p > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}$, $p = 0$ on $\overline{T}_\tau$ and $0 < p \leqslant b$ on $\overline{T}_\delta \setminus \overline{T}_\tau$. We choose $\tau \in (0,\delta)$ such that a is less than the first Dirichlet eigenvalue of $(-\Delta)$ in T_τ. Let u^* be the unique positive solution of $\Delta u + au = p(x)f(u)$ in C_δ, $u|_{\partial C_\delta} = 1$. Define $0 < u^+ \in C^2(\overline{\Omega} \setminus D_0)$ such that $u^+ = v_\infty$ on $\Omega \setminus (T_\delta \cup D_0)$ and $u^+ = 1$ (resp., $u^+ = u^*$) on $\overline{T}_{\delta/2}$ if $\mathcal{B} = \mathcal{R}$ (resp., $\mathcal{B} = \mathcal{D}$). Note that $\tilde{u} = \xi u^+$ is a supersolution of (13) if $\xi > 1$ is large. Clearly, $\tilde{u} = \infty$ on ∂D_0 and $\mathcal{B}\tilde{u} \geqslant 0$ on $\partial\Omega$. By (A_1), $\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) \leqslant 0$ on $\Omega \setminus (T_\delta \cup D_0)$, $\forall\xi > 1$. If $\mathcal{B} = \mathcal{D}$ then $\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) \leqslant \xi\Delta u^* + a\xi u^* - p(x)f(\xi u^*) \leqslant 0$ on $T_{\delta/2}$, $\forall\xi > 1$. If $\mathcal{B} = \mathcal{R}$ (resp., $\mathcal{B} = \mathcal{D}$) then $\min_{\overline{T}_\delta} b > 0$ (resp., $\inf_{T_\delta\setminus T_{\delta/2}} b > 0$). For $\xi > 1$ large, $\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) = \xi(\Delta u^+ + au^+ - b(x)f(\xi u^+)/\xi) \leqslant 0$ on T_δ (resp., $T_\delta \setminus T_{\delta/2}$) when $\mathcal{B} = \mathcal{R}$ (resp., $\mathcal{B} = \mathcal{D}$). The sub-supersolutions method and [1, Corollary A.2] yield the existence of a positive solution of (13). The uniqueness follows by Lemma 2.2. □

Proof of Theorem 1.1 concluded. If (1) has a nonnegative solution then, by the strong maximum principle, it is positive. By the assumption $\overline{\Omega_{0,b} \setminus D_0} \subset \Omega \setminus D_0$ and [1, Lemma 3.2], we get $a < \lambda_{\infty,1}(D_1)$.

Fix $a < \lambda_{\infty,1}(D_1)$ and let u_n $(n \geqslant 1)$ be the unique positive solution of (13) with $\Phi \equiv n$. By Lemma 2.2, $u_n \leqslant u_{n+1} \leqslant \tilde{u}$ on $\overline{\Omega} \setminus D_0$. Thus (u_n) converges to the *minimal* positive solution of (1).

Define $\Omega_m = \{x \in \Omega\colon d(x) \leqslant 1/m\}$ for $m \geqslant m_1$, where $m_1 > 0$ is large so that $b > 0$ on $\underline{\Omega_{m_1} \setminus D_0}$. Let v_m be the minimal positive solution of (1) with D_0 replaced by Ω_m. By Lemma 2.2, $v_m \geqslant v_{m+1} \geqslant u$ on $\overline{\Omega} \setminus \Omega_m$ where u

is any positive solution of (1). This, together with a regularity and compactness argument, shows that the pointwise limit of (v_m) is the maximal positive solution of (1). □

Proof of Theorem 1.3 concluded. Fix $a < \lambda_{\infty,1}(D_1)$. By [2, Remark 2], (A_2) is fulfilled. By Theorem 1.1, there exists at least a positive solution of (1). We now prove that (2) holds for any positive solution of (1). Fix $\varepsilon \in (0, c/2)$. Let $\delta > 0$ be small such that (i) $\operatorname{dist}(x, \partial D_0)$ is a C^2-function on the set $\{x \in \mathbb{R}^N\colon \operatorname{dist}(x, \partial D_0) < 2\delta\}$, (ii) k is nondecreasing on $(0, 2\delta)$, (iii) $b(x)/k^2(d(x)) \in (c-\varepsilon, c+\varepsilon)$, $\forall x \in \Omega$ with $d(x) \in (0, 2\delta)$ and (iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (see [2]). Let $\sigma \in (0, \delta)$ be arbitrary. Set $\xi^{\pm} = [(2+\ell_1\rho)/(c \mp 2\varepsilon)(2+\rho)]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x)+\sigma)\xi^-$, $\forall x$ with $\sigma < d(x) + \sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x) - \sigma)\xi^+$, $\forall x$ with $d(x) \in (\sigma, 2\delta)$. As in [2], we can assume $\Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \leqslant 0$, $\forall x$ with $\sigma < d(x) < 2\delta$ and $\Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \geqslant 0$ $\forall x \in \Omega \setminus D_0$ with $d(x) + \sigma < 2\delta$.

Define $\Omega_\delta = \{x \in \Omega\colon d(x) < \delta\}$. Let $\omega \subset\subset D_0$ be such that a is less than the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\tilde{D} := \operatorname{int}(D_0 \setminus \omega)$. Let $p \in C^{0,\mu}(\overline{\Omega}_\delta)$ satisfy $0 < p \leqslant b$ on $\overline{\Omega}_\delta \setminus D_0$, $p \equiv 0$ on $D_0 \setminus \omega$ and $p > 0$ on ω. By [1, Theorem 1.1], there exists a positive boundary blow-up solution to $\Delta w + a w = p(x) f(w)$ in Ω_δ. Let u_a be an arbitrary solution of (1) and let $v := u_a + w$. Then v satisfies $\Delta v + a v - b(x) f(v) \leqslant 0$ in $\Omega_\delta \setminus D_0$. Lemma 2.2 yields $u_a + w \geqslant v_\sigma^-$ on $\Omega_\delta \setminus D_0$. Similarly, $v_\sigma^+ + w \geqslant u_a$ on $\Omega_\delta \setminus \overline{\Omega}_\sigma$. Letting $\sigma \to 0$, we find that $h(d)\xi^+ + 2w \geqslant u_a + w \geqslant h(d)\xi^-$ on $\Omega_\delta \setminus D_0$. It follows that $\xi^- \leqslant \liminf_{d(x)\searrow 0} u_a(x)/h(d(x)) \leqslant \limsup_{d(x)\searrow 0} u_a(x)/h(d(x)) \leqslant \xi^+$. Letting $\varepsilon \to 0$ we arrive at (2).

Let u_1 and u_2 be two arbitrary positive solutions of (1). For any $\varepsilon > 0$, define $\tilde{u}_i = (1+\varepsilon)u_i$, $i = 1, 2$. By (2), we deduce $\lim_{d(x)\searrow 0}[u_1(x) - \tilde{u}_2(x)] = \lim_{d(x)\searrow 0}[u_2(x) - \tilde{u}_1(x)] = -\infty$. Using (A_1), we find $\Delta \tilde{u}_i \leqslant b(x) f(\tilde{u}_i) - a\tilde{u}_i$ on $\Omega \setminus D_0$. Since $\mathcal{B}\tilde{u}_i = \mathcal{B}u_i = 0$ on $\partial\Omega$, by Lemma 2.2 we find $u_1 \leqslant \tilde{u}_2$ resp., $u_2 \leqslant \tilde{u}_1$ on $\overline{\Omega} \setminus D_0$. Letting $\varepsilon \to 0$, we conclude that $u_1 \equiv u_2$.

Define $U_1(u) = 0$ for $u \leqslant 0$, $U_1(u) = 1/\int_u^\infty [2F(s)]^{-1/2}\,\mathrm{d}s$ for $u > 0$ and $U_2(u) = 1/\int_0^{1/u} k(s)\mathrm{d}s$ for $u > 1/\nu$. We see that $U_1 : (0,\infty) \to (0,\infty)$ is a C^1-increasing and bijective function. Thus, for each $y > 0$, $U_1^{\leftarrow}(y) = \inf\{s\colon U_1(s) \geqslant y\}$ coincides with the inverse function of U_1 at y. Hence, $h(1/u) = U_1^{\leftarrow}(U_2(u))$ for $u > 1/\nu$. Clearly, $\lim_{u\to\infty} U_1(u) = \lim_{u\to\infty} U_2(u) = \infty$ and $U_1(u) \in \mathbb{R}_{\rho/2}$.

Suppose $\ell_1 \neq 0$. By [3, Proposition 2], $k(1/u) \in \mathbb{R}_{(\ell_1-1)/\ell_1}$. Thus, $U_2(u) \in \mathbb{R}_{1/\ell_1}$. Using [6, Proposition 0.8 (iv) and (v)], we obtain $U_1^{\leftarrow}(u) \in \mathbb{R}_{2/\rho}$ and $U_1^{\leftarrow} \circ U_2 \in \mathbb{R}_{2/(\rho\ell_1)}$. This proves (3).

Assume $\ell_1 = 0$. Then $U_2(u) = d_0^{-1} \exp\{\int_{d_1}^u [s\Lambda(s)]^{-1}\,\mathrm{d}s\}$ for $u \geqslant d_1$, where $0 < \Lambda \in C^1[d_1, \infty)$ satisfies $\lim_{u\to\infty} \Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0$. So, $U_2(u)$ is Γ-varying at $u = \infty$ with auxiliary function $u\Lambda(u)$ (see [6, p. 106]). Since $U_1^{\leftarrow}(u)$ is monotone on $(0,\infty)$ and $U_1^{\leftarrow}(u) \in \mathbb{R}_{2/\rho}$, we infer that $h(1/u)$ is Γ-varying at $u = \infty$ with auxiliary function $\rho u\Lambda(u)/2$ (see [6, p. 36]). If, in addition, $\Lambda \in \mathbb{R}_j$ $(j \leqslant 0)$ then $W(u) := \ln U_2(u) \in \mathbb{R}_{-j}$. Letting $T(u) = 1/[\xi_0 U_1^{\leftarrow}(u)]$ for $u > 0$, we conclude (4). □

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