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Global gradient bounds for dissipative diffusion operators

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Abstract

Let L be a second order elliptic operator on $\mathbb{R}^d$ with a constant diffusion matrix and a dissipative (in a weak sense) drift $b \in L^p_{\mathrm{loc}}$ with some $p > d$. We assume that L possesses a Lyapunov function, but no local boundedness of b is assumed. It is known that then there exists a unique probability measure μ satisfying the equation $L^*\mu = 0$ and that the closure of L in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geqslant 0}$ with the resolvent $\{G_\lambda\}_{\lambda>0}$. We prove that, for any Lipschitzian function $f \in L^1(\mu)$ and all $t, \lambda > 0$, the functions $T_t f$ and $G_\lambda f$ are Lipschitzian and $\sup_{x,t} |\nabla T_t f(x)| \leqslant \sup_x |\nabla f(x)|$ and $\sup_x |\nabla G_\lambda f(x)| \leqslant \frac{1}{\lambda} \sup_x |\nabla f(x)|$. In addition, we show that for every bounded Lipschitzian function g, the function $G_\lambda g$ is the unique bounded solution of the equation $\lambda f - Lf = g$ in the Sobolev class $H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$. ***To cite this article: V.I. Bogachev et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Estimations globales de gradient pour les opérateurs de diffusion. Soit L un opérateur elliptique sur $\mathbb{R}^d$ tel que son terme du premier ordre $b \in L^p_{\mathrm{loc}}$, $p > d$, soit dissipatif (mais pas nécessairement localement borné) et qu'il existe une fonction de Liapounoff. Il est connu qu'il existe une probabilité unique μ telle que $L^*\mu = 0$ au sens faible et la fermeture de L dans $L^1(\mu)$ est le générateur d'un semigroupe markovien $\{T_t\}_{t\geqslant 0}$ de résolvante $\{G_\lambda\}_{\lambda>0}$. Nous montrons que pour chaque fonction lipschitzienne $f \in L^1(\mu)$ et tous $t, \lambda > 0$ les fonctions $T_t f$ et $G_\lambda f$ sont lipschitziennes et on a $\sup_{x,t} |\nabla T_t f(x)| \leqslant \sup_x |\nabla f(x)|$ et $\sup_x |\nabla G_\lambda f(x)| \leqslant \frac{1}{\lambda} \sup_x |\nabla f(x)|$. De plus, nous montrons que pour chaque fonction bornée lipschitzienne g la fonction $G_\lambda g$ est la solution unique bornée de l'équation $\lambda f - Lf = g$ dans la classe de Sobolev $H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$. ***Pour citer cet article : V.I. Bogachev et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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Soit L un opérateur elliptique sur $\mathbb{R}^d$ de la forme

$$Lf=\sum_{i,j\leqslant d}a^{ij}\partial_{x_i}\partial_{x_j}f+\sum_{i\leqslant d}b^i\partial_{x_i}f,$$

où $A=(a^{ij})$ est une matrice constante symétrique positive et $b=(b^i):\mathbb{R}^d\to\mathbb{R}^d$ est une application mesurable telle que $|b|\in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ avec $p>d$. Nous considérons l'équation

$$\lambda f-Lf=g,\quad \lambda>0,$$

et obtenons une condition qui fournit l'inégalité

$$\sup_x\bigl|\nabla f(x)\bigr|\leqslant\lambda^{-1}\sup_x\bigl|\nabla g(x)\bigr|$$

pour chaque fonction bornée lipschitzienne g. Cette inégalité est connue (par exemple, elle a été établie dans [8,9] même en dimension infinie) sous les hypothèses suivantes : $(b(x)-b(y),x-y)\leqslant-\omega|x-y|^2$, oú $\omega>0$, et b est lipschitzienne (ou continue et $|b(x)||x|^2\in L^2(\mu)$). Ces conditions sont assez réstrictives pour applications en analyse stochastique. Supposons que l'application b soit dissipative au sens plus faible : pour chaque $h\in\mathbb{R}^d$ on a $(b(x+h)-b(x),h)\leqslant 0$ p.p. Suppposons aussi qu'il existe une fonction $V\geqslant 0$ de la classe C^2 (une fonction de Liapounoff) telle que $\lim_{|x|\to\infty}V(x)=\infty$ et $\lim_{|x|\to\infty}LV(x)=-\infty$. Par exemple, si $(b(x),x)\leqslant c<0$ en dehors d'une boule, on peut prendre $V(x)=(x,x)^m$ avec m suffisamment grand. D'après [5], il existe une mesure de probabilité μ telle que $L^*\mu=0$ au sens suivant :

$$\int_{\mathbb{R}^d}L\varphi\,\mathrm{d}\mu=0\quad\forall\varphi\in C_0^\infty\bigl(\mathbb{R}^d\bigr).$$

La mesure μ possède une densité continue positive ϱ de la classe $H^{p,1}_{\mathrm{loc}}(\mathbb{R}^d)$. De plus, il existe un unique semigroupe fortement continu markovien $(T_t)_{t\geqslant 0}$ dans $L^1(\mu)$ tel que μ soit invariante pour $(T_t)_{t\geqslant 0}$, c'est-à-dire que

$$\int_{\mathbb{R}^d}T_tf\,\mathrm{d}\mu=\int_{\mathbb{R}^d}f\,\mathrm{d}\mu\quad\forall f\in L^1(\mu),$$

et le générateur $\bar{L}$ de ce semigroupe est une extention de $(L,C_0^\infty(\mathbb{R}^d))$. Soit G_λ la résolvante correspondante. Notons $D(\bar{L})$ le domaine de $\bar{L}$ dans $L^1(\mu)$. Étant donné $\lambda>0$, pour chaque $g\in L^1(\mu)$ il existe une fonction unique $f\in\bar{L}$ telle que $\lambda f-\bar{L}f=g$. Si $g\in L^2(\mu)$, on a $f\in L^2(\mu)$ et $f\in H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$, et notre équation s'écrit comme $\lambda f-Lf=g$ p.p.

Théorème 0.1. *Pour chaque fonction lipschitzienne $f\in L^1(\mu)$ et tous $t,\lambda>0$ les fonctions T_tf et $G_\lambda f$ sont lipschitziennes et on a*

$$\sup_{x,t}\bigl|\nabla T_tf(x)\bigr|\leqslant\sup_x\bigl|\nabla f(x)\bigr|,\qquad\sup_x\bigl|\nabla G_\lambda f(x)\bigr|\leqslant\frac{1}{\lambda}\sup_x\bigl|\nabla f(x)\bigr|.$$

De plus, étant donnée une fonction bornée lipschitzienne g, la fonction $G_\lambda g$ est la solution unique bornée de l'équation $\lambda f-Lf=g$ dans la classe de Sobolev $H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$.

1. Introduction and main result

Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$Lu = \sum_{i,j\leqslant d} a^{ij}\partial_{x_i}\partial_{x_j}u + \sum_{i\leqslant d} b^i\partial_{x_i}u, \tag{1}$$

where $A = (a^{ij})$ is a constant strictly positive definite symmetric matrix and $b = (b^i)$ is a measurable vector field which satisfy the following hypotheses:

(Ha) $|b| \in L^p_{\rm loc}(\mathbb{R}^d)$ with some $p > d$, $p \geqslant 2$;
(Hb) b is *dissipative* i.e., for every $h \in \mathbb{R}^d$, there exists a measure zero set $N_h \subset \mathbb{R}^d$ such that $(b(x+h)-b(x), h) \leqslant 0$ for all $x \in \mathbb{R}^d \setminus N_h$;
(Hc) there exists a *Lyapunov function* V for L, i.e., a nonnegative C^2-function V such that $V(x) \to +\infty$ and $LV(x) \to -\infty$ as $|x| \to \infty$.

Hypotheses (Ha) and (Hc) imply (see [1] and [5]) that there exists a unique probability measure μ on $\mathbb{R}^d$ such that μ has a strictly positive continuous weakly differentiable density ϱ, $|\nabla\varrho| \in L^p_{\rm loc}(\mathbb{R}^d)$, and $L^*\mu = 0$ in the following weak sense: $\int Lu\,\mathrm{d}\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. The closure $\bar{L}$ of L with domain $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geqslant 0}$ for which μ is invariant. Let $D(\bar{L})$ denote the domain of $\bar{L}$ in $L^1(\mu)$ and let $\{G_\lambda\}_{\lambda>0}$ denote the corresponding resolvent, i.e., $G_\lambda = (\lambda - \bar{L})^{-1}$. The restrictions of T_t and G_λ to $L^2(\mu)$ are contractions on $L^2(\mu)$. In particular, if $v \in D(\bar{L})$ is such that $\lambda v - \bar{L}v = g \in L^2(\mu)$, then $v \in L^2(\mu)$. Moreover, it follows by [5, Theorem 2.8] (for bounded g this follows also from [11, Lemma 2.1]) that one has $v \in H^{2,2}_{\rm loc}(\mathbb{R}^d)$ and $\bar{L}v = Lv$ a.e., so that one has a.e.

$$\lambda v - Lv = g. \tag{2}$$

In fact, due to our assumptions on the coefficients of L one has even $v \in H^{p,2}_{\rm loc}(\mathbb{R}^d)$ (see [7]). The main result of this Note is the following theorem.

Theorem 1.1. *Let A and b satisfy* (Ha), (Hb) *and* (Hc). *Then, for any Lipschitzian function $f \in L^1(\mu)$ and all $t, \lambda > 0$, $T_t f$ and $G_\lambda f$ have Lipschitzian versions such that*

$$\sup_{x,t}\bigl|\nabla T_t f(x)\bigr| \leqslant \sup_x\bigl|\nabla f(x)\bigr| \quad \text{and} \quad \sup_x\bigl|\nabla G_\lambda f(x)\bigr| \leqslant \frac{1}{\lambda}\sup_x\bigl|\nabla f(x)\bigr|. \tag{3}$$

In addition, for every bounded Lipschitzian function g on $\mathbb{R}^d$ and any $\lambda > 0$, the function $G_\lambda g$ is the unique bounded solution of Eq. (2) *in the Sobolev class $H^{2,2}_{\rm loc}(\mathbb{R}^d)$.*

Such estimates have been established by probabilistic methods in [9] (even in the infinite dimensional case) under the assumption that $(b(x) - b(y), x - y) \leqslant -\alpha|x - y|^2$ (*strong dissipativity*) with some $\alpha > 0$, b is m-dissipative and one has $(1 + |b|^2)(1 + |x|^4) \in L^1(\mu)$; see also [8] for the case where the last assumption is replaced by the one that b is globally Lipschitzian. Note that strong dissipativity implies (Hc) with $V(x) = |x|^2$. In fact a weaker assumption $(b(x), x) \leqslant c < 0$ outside a ball, implies that one can take $V(x) = |x|^m$ for m big enough.

Let $\hat{L}$ be the elliptic operator with the same second order part as L, but with drift is $\hat{b} = 2A\nabla\varrho/\varrho - b$. Then by the integration by parts formula

$$\int \psi L\varphi\,\mathrm{d}\mu = \int \varphi\hat{L}\psi\,\mathrm{d}\mu \quad \text{for all } \psi, \varphi \in C_0^\infty\bigl(\mathbb{R}^d\bigr).$$

In addition, for any $\lambda > 0$, the ranges of $\lambda - L$ and $\lambda - \hat{L}$ on $C_0^\infty(\mathbb{R}^d)$ are dense in $L^1(\mu)$. The operator $\hat{L}$ also generates a Markov semigroup on $L^1(\mu)$ with respect to which μ is invariant. The corresponding resolvent is denoted by $\widehat{G}_\lambda$. For the proofs we refer to [4, Proposition 2.9] or [12, Proposition 1.10(b)] (see also [5, Theorem 3.1]).

Lemma 1.2. *Suppose that b is infinitely differentiable, Lipschitzian, and strongly dissipative, so for some $\alpha > 0$, one has $(b(x+h) - b(x), h) \leqslant -\alpha(h,h)$ for all $x, h \in \mathbb{R}^d$. Let $\lambda > 0$ and let $v \in L^2(\mu)$ satisfy the inequality $(\lambda - L)v \leqslant 0$ in the μ-weak sense, i.e., $\int v(\lambda - \hat{L})\varphi\,\mathrm{d}\mu \leqslant 0$ for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then $v \leqslant 0$.*

Proof. Letting $V(x) = (x,x)$ and denoting the matrix trace by tr, we obtain

$$LV(x) = 2\,\mathrm{tr}\,A + 2\big(b(x), x\big) \leqslant 2\,\mathrm{tr}\,A - 2\alpha(x,x) + 2\big(b(0), x\big) \leqslant 2\,\mathrm{tr}\,A + \alpha^{-1}\big|b(0)\big|^2 - \alpha(x,x).$$

According to [1], μ has all moments, hence $|b| \in L^2(\mu)$. As shown in [2], this implies $|\nabla\varrho/\varrho| \in L^2(\mu)$. Let $\zeta_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \leqslant \zeta_0 \leqslant 1$ and $\zeta_0(x) = 1$ whenever $|x| \leqslant 1$. Let $\zeta_k(x) = \zeta_0(x/k)$, $k \in \mathbb{N}$. Then $0 \leqslant \zeta_k \leqslant 1$, $|\nabla\zeta_k|$ is bounded uniformly in k, $\zeta_k \to 1$ pointwise and $|\nabla\zeta_k|, L\zeta_k, \hat{L}\zeta_k \to 0$ in $L^2(\mu)$ as $k \to \infty$.

Let $\eta \in C_0^\infty(\mathbb{R}^d)$, $\eta \geqslant 0$ and $u := \widehat{G}_\lambda\eta$. Then u is bounded nonnegative, by the Markovian property, and smooth, by the elliptic regularity, since $\hat{b}$ is smooth. It is known that $|\nabla u| \in L^2(\mu)$. This follows by [12, Theorem 1.5(c)] or by [11, Lemma 2.1]), but can be verified directly as follows. For all $\zeta \in C_0^\infty(\mathbb{R}^d)$ one has $\int u(\lambda u - \hat{L}u)\zeta\,\mathrm{d}\mu = \int \zeta u\eta\,\mathrm{d}\mu$. Since $u\hat{L}u = \frac{1}{2}\hat{L}u - (A\nabla u, \nabla u)$, it follows that

$$\int u^2\left(\lambda - \frac{1}{2}L\right)\zeta\,\mathrm{d}\mu + \int (A\nabla u, \nabla u)\zeta\,\mathrm{d}\mu = \int u\eta\zeta\,\mathrm{d}\mu.$$

Choosing ζ_k as defined above, we get $|\nabla u| \in L^2(\mu)$.

Let now $\varphi_k := \zeta_k u$. Then $\varphi_k \in C_0^\infty(\mathbb{R}^d)$, $\varphi_k \geqslant 0$ and

$$(\lambda - \hat{L})\varphi_k = \zeta_k\eta + u\,\hat{L}\zeta_k + 2(A\nabla\zeta_k, \nabla u) \to \eta \quad \text{in } L^2(\mu) \text{ as } k \to \infty$$

by the dominated convergence theorem, since u is bounded and $|\nabla u| \in L^2(\mu)$. Hence

$$\int v\eta\,\mathrm{d}\mu = \lim_{k\to\infty}\int v(\lambda - \hat{L})\varphi_k\,\mathrm{d}\mu \leqslant 0,$$

which yields that $v \leqslant 0$, since η is an arbitrary smooth function with compact support. □

In the following lemma we also consider the case of smooth, Lipschitzian, strongly dissipative b. This lemma follows, of course, from [8,9], where probabilistic arguments are given, but for the reader's convenience we include an alternative analytic proof.

Lemma 1.3. *Let b be the same as in the previous lemma. Then, for any $\lambda > 0$ and any smooth bounded Lipschitzian function f, one has pointwise*

$$|\nabla G_\lambda f| \leqslant G_\lambda|\nabla f|.$$

In particular, $\sup_x |\nabla G_\lambda f(x)| \leqslant \lambda^{-1}\sup_x |\nabla f(x)|$.

Proof. Let $u = G_\lambda f$. As explained in the proof of the previous lemma, $u \in C^\infty(\mathbb{R}^d)$ and $|\nabla u| \in L^2(\mu)$. Observe that $(\lambda - \bar{L})G_\lambda|\nabla f| = (\lambda - L)G_\lambda|\nabla f| = |\nabla f|$, because $|\nabla f|$ is bounded Lipschitzian and $G_\lambda|\nabla f| \in H_{\mathrm{loc}}^{p,2}(\mathbb{R}^d)$. Hence it suffices to show that $v := |\nabla u|$ is a μ-weak sub-solution of the equation $(\lambda - L)v = |\nabla f|$, i.e.,

$$\int v(\lambda\varphi - \hat{L}\varphi)\,\mathrm{d}\mu \leqslant \int |\nabla f|\varphi\,\mathrm{d}\mu \tag{4}$$

for every nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d)$, because then Lemma 1.2 implies the assertion. Let $\varepsilon > 0$ and $v_\varepsilon := (|\nabla u|^2 + \varepsilon)^{1/2}$. It is readily verified that v_ε satisfies the equation $\lambda v_\varepsilon - L v_\varepsilon - w_\varepsilon v_\varepsilon = f_\varepsilon$, where

$$w_\varepsilon := (Db\nabla u, \nabla u) v_\varepsilon^{-2} \leqslant 0,$$

$$f_\varepsilon := \lambda\varepsilon v_\varepsilon^{-1} + (\nabla u, \nabla f) v_\varepsilon^{-1} - v_\varepsilon^{-1}\big[\mathrm{tr}(D^2 u A D^2 u) - v_\varepsilon^{-2}(\nabla u, D^2 u A D^2 u \nabla u)\big].$$

Let $\xi := v_\varepsilon^{-1}\nabla u$ and $S := D^2 u$. Noting that $|\xi| \leqslant 1$ and the matrix SAS is symmetric and nonnegative definite, we obtain $(\xi, \nabla f) \leqslant |\nabla f|$ and $\mathrm{tr}(SAS) - (\xi, SAS\xi) \geqslant 0$. Therefore, $(\lambda - L)v_\varepsilon \leqslant \lambda\varepsilon v_\varepsilon^{-1} + |\nabla f|$ pointwise, in particular, in the μ-weak sense. Letting $\varepsilon \to 0$ we obtain (4). □

2. Proof of Theorem 1.1

We recall that if a sequence of functions on $\mathbb{R}^d$ is uniformly Lipschitzian with constant L and bounded at a point, then it contains a subsequence that converges uniformly on every ball to a function that is Lipschitzian with the same constant. Therefore, approximating f in $L^1(\mu)$ by a sequence of bounded smooth functions f_j with $\sup_x |\nabla f_j(x)| \leqslant \sup_x |\nabla f(x)|$, it suffices to prove our estimates for smooth bounded f. Moreover, due to Euler's formula $T_t f = \lim_n (\frac{t}{n} G_{t/n})^n f$, it suffices to establish the resolvent estimate. First we construct a suitable sequence of smooth strongly dissipative Lipschitzian vector fields b_k such that $b_k \to b$ in $L^p(U)$ on every ball U as $k \to \infty$. Let $\sigma_j(x) = j^{-d}\sigma(x/j)$, where σ is a smooth compactly supported probability density. Let $\beta_j := b * \sigma_j$. Then β_j is smooth and dissipative and $\beta_j \to b$, $j \to \infty$, in $L^p(U)$ on every ball U. For every $\alpha > 0$, the mapping $I - \alpha\beta_j$ is a homeomorphism of $\mathbb{R}^d$ and the inverse mapping $(I - \alpha\beta_j)^{-1}$ is Lipschitzian with constant α^{-1} (see [6]). Let us consider the Yosida approximations

$$F_\alpha(\beta_j) := \alpha^{-1}\big((I - \alpha\beta_j)^{-1} - I\big) = \beta_j \circ (I - \alpha\beta_j)^{-1}.$$

It is known (see [6, Chapter II]) that $|F_\alpha(\beta_j)(x)| \leqslant |\beta_j(x)|$, the mappings $F_\alpha(\beta_j)$ converge locally uniformly to β_j as $\alpha \to 0$, and one has $(F_\alpha(\beta_j)(x) - F_\alpha(\beta_j)(y), x - y) \leqslant 0$.

Thus, the sequence $b_k := F_{1/k}(b * \sigma_k) - \frac{1}{k} I$, $k \in \mathbb{N}$, is the desired one. For every $k \in \mathbb{N}$, let L_k be the elliptic operator defined by (1) with the same constant matrix A and drift b_k in place of b. Let $\mu_k = \varrho_k\,\mathrm{d}x$ be the corresponding invariant probability measure and let $G_\lambda^{(k)}$ denote the associated resolvent family on $L^1(\mu_k)$. Since b_k is smooth, Lipschitzian and strongly dissipative, $v_k := G_\lambda^{(k)} f$ is smooth, bounded, Lipschitzian and

$$\sup_x \big|v_k(x)\big| \leqslant \frac{1}{\lambda} \sup_x \big|f(x)\big| \quad \text{and} \quad \sup_x \big|\nabla v_k(x)\big| \leqslant \frac{1}{\lambda} \sup_x \big|\nabla f(x)\big|$$

by Lemma 1.3. Moreover, for every ball $U \subset \mathbb{R}^d$, the functions v_k are uniformly bounded in the Sobolev space $H^{2,2}(U)$, since the mappings $|b_k|$ are bounded in $L^p(U)$ uniformly in k and f is bounded. Thus, the sequence $\{v_k\}$ contains a subsequence, again denoted by $\{v_k\}$, that converges locally uniformly to a bounded Lipschitzian function $v \in H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\sup_x \big|v(x)\big| \leqslant \lambda^{-1} \sup_x \big|f(x)\big| \quad \text{and} \quad \sup_x \big|\nabla v(x)\big| \leqslant \lambda^{-1} \sup_x \big|\nabla f(x)\big|,$$

and, in addition, the restrictions of v_k to any ball U converge to $v|_U$ weakly in $H^{2,2}(U)$. Now we show that $v = G_\lambda f$. Note that $\varrho_k \to \varrho$ uniformly on balls according to [1,3]. Hence, given $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in a ball U, we have

$$\int [\lambda v - Lv - f]\varphi\varrho\,\mathrm{d}x = \lim_{k\to\infty} \int [\lambda v_k - L_k v_k - f]\varphi\varrho_k\,\mathrm{d}x = 0$$

by weak convergence of v_k to v in $H^{2,2}(U)$ combined with convergence of b_k to b in $L^p(U, \mathbb{R}^d)$. Therefore, by the integration by parts formula

$$\int v(\lambda\varphi - \hat{L}\varphi)\,\mathrm{d}\mu = \int f\varphi\,\mathrm{d}\mu$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The function $G_\lambda f$ is bounded and satisfies the same relation, so it remains to recall that if a bounded function u satisfies $\int u(\lambda\varphi - \hat{L}\varphi)\,\mathrm{d}\mu = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $u = 0$ a.e., since $(\lambda - \hat{L})(C_0^\infty(\mathbb{R}^d))$ is dense in $L^1(\mu)$. The same reasoning proves also the last claim. □

Remark 1. Apart from weaker assumptions on b, the main novelty of the result is uniqueness in the class of bounded solutions of the class $H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$ that are not supposed in advance to be in the domain of generator $\bar{L}$. We emphasize that the assumption of existence of a Lyapunov function has been only used in the uniqueness statement. Our reasoning without that assumption (i.e., only with (Ha) and (Hb)) shows that given a bounded Lipschitzian function g, there exists some bounded solution $v \in H^{2,2}_{\mathrm{loc}}(\mathbb{R}^d)$ of the equation $(\lambda - L)v = g$ satisfying (3). Indeed, we obtain uniformly bounded and uniformly Lipschitzian functions $v_k = G_\lambda^{(k)}$ satisfying the equations $\lambda v_k - L_k v_k = g$. Then we find a subsequence in $\{v_k\}$ that converges weakly in $H^{2,2}(U)$ for every ball U. The limit is a desired solution. We do not know whether such a solution is unique in this case. Note also that if b is locally Hölder continuous, then, by the classical theory (see, e.g., [10]), the second derivative of f is locally Hölder continuous.

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