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Homogenization near resonances and artificial magnetism from dielectrics

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Abstract

The homogenization of periodic dielectric structures in harmonic regime usually leads to an effective permittivity tensor ε^{eff}. It has been observed by Bouchitté and Felbacq [Waves Random Media 7 (1997) 245–256], that in the high contrast case (high conductivity fibers), this tensor depends on the angular frequency ω. In this Note, we enlight a new effect induced by microscopic resonances which leads in parallel to a possibly negative effective permeability $\mu^{\text{eff}}(\omega)$ (although the original medium is assumed to be nonmagnetic i.e. $\mu = 1$). ***To cite this article: G. Bouchitté, D. Felbacq, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Homogénéisation de diélectriques près des résonances et magnétisme artificiel. L'homogénéisation de structures diélectriques périodiques en régime harmonique fait apparaitre en général un tenseur de permittivité effective ε^{eff}. Il a été remarqué par Bouchitté et Felbacq [Waves Random Media 7 (1997) 245–256] que dans le cas de forts contrastes (fibres de grande conductivité), ce tenseur peut dépendre de la fréquence ω. Dans cette note, nous mettons en évidence un effet nouveau dû à des micro-résonances et qui conduit, malgré l'absence initiale de propriétés magnétiques (i.e. $\mu = 1$), à une perméabilité effective $\mu^{\text{eff}}(\omega)$ qui peut éventuellement être négative. ***Pour citer cet article : G. Bouchitté, D. Felbacq, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

On considère le problème de la diffraction d'une onde monochromatique par une structure composée d'une matrice diélectrique contenue dans un cylindre $\mathcal{O} = \Omega \times \mathbb{R}$, où Ω est un domaine de $\mathbb{R}^2$ et dans laquelle sont plongées des tiges parallèles, infiniment longues et de très forte permittivité. Les sections de ces tiges sont réparties ηY périodiquement dans Ω où η est un petit paramètre donné et $Y = [-1/2, 1/2[^2$. Elles occupent un domaine

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$D_\eta := \bigcup_{i\in I_\eta} \eta(i+D)$ où $D \Subset Y$ est un ouvert régulier connexe ainsi que son complémentaire $Y^* := Y \setminus D$ et où $I_\eta = \{i \in \mathbb{Z}^2\colon\ \eta(i+D) \Subset \Omega\}$. La structure diffractante dont la perméabilité est supposée constante est caractérisée par sa permittivité (relative) $\varepsilon_\eta = a_\eta^{-1}$ où a_η est la fonction définie sur $\mathbb{R}^2$ par :

$$a_\eta(x) := \frac{\eta^2}{\varepsilon_i} \quad \text{si } x \in D_\eta, \qquad a_\eta(x) := \frac{1}{\varepsilon_e} \quad \text{si } x \in \Omega \setminus D_\eta, \qquad a_\eta(x) := 1 \quad \text{si } x \in \mathbb{R}^2 \setminus \Omega.$$

La dépendance temporelle étant en $\mathrm{e}^{-\mathrm{i}\omega t}$, on supposera que ε_e est un réel positif (diélectrique sans pertes) et que la constante ε_i est un complexe de partie imaginaire $\geqslant 0$. Le facteur en η^2 choisi dans les fibres correspond à un diamètre optique constant pour chacun d'entre elles. Le nombre d'onde k est défini par $k = \omega/c$, où c est la vitesse de la lumière dans le vide. La structure est éclairée par une onde incidente u^i polarisée linéairement en $H_\|$ de sorte que le champ magnétique total $u_\eta(x_1, x_2)$ vérifie :

$$\operatorname{div}(a_\eta \nabla u_\eta) + k^2 u_\eta = 0 \quad \text{sur } \mathbb{R}^2, \qquad u_\eta - u^i \text{ satisfait la condition d'onde sortante (3).}$$

On introduit, pour tout $\omega > 0$, la solution M_ω dans $H^1(D)$ (quand elle existe) du problème :

$$\Delta M_\omega + k^2 \varepsilon_i M_\omega = 0 \quad \text{sur } D, \qquad M_\omega = 1 \quad \text{sur } \partial D$$

et on pose (voir (6) et (7) pour des formes explicites) :

$$\mu^{\mathrm{eff}}(\omega) = |Y^*| + \int_D M_\omega \,\mathrm{d}y.$$

Théorème 0.1. *Supposons que ω ne prend pas un certain ensemble discret de valeurs* (*condition* (5))*. Alors la suite $\{u_\eta\}$ reste bornée dans $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ et converge à 2 échelles vers $u_0(x,y) := u(x)\, M_\omega(y)$ où M_ω est déterminée par* (6) *et u est l'unique solution dans $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ du système*

$$\begin{cases} \operatorname{div}(A^{\mathrm{eff}} \nabla u) + k^2 \mu^{\mathrm{eff}}(\omega) u = 0 & \textit{sur } \Omega, \\ \Delta u + k^2 u = 0 & \textit{sur } \mathbb{R}^2 \setminus \Omega, \\ (\nabla u)^+ \cdot n = (A^{\mathrm{eff}} \nabla u)^- \cdot n, \quad u^+ = u^- & \textit{sur } \partial\Omega, \\ u - u^i \ \textit{vérifie } (3), \end{cases}$$

où $\mu^{\mathrm{eff}}(\omega)$, A^{eff} sont donnés par (7), (8)*, et les indices $\pm$ indiquent les traces extérieures et intérieures.*

Il est important de noter que, d'après (7), la partie réelle de $\mu^{\mathrm{eff}}(\omega)$ est négative sur une réunion dénombrable d'intervalles. Cela correspond à des bandes de fréquence où les ondes ne peuvent pas se propager dans Ω (bandes interdites photoniques). Cet effet important est lié à des résonances internes. Il peut avoir lieu sans aucune dissipation d'énergie (i.e. avec ε_i et ε_e réels positifs).

1. Introduction

The diffractive properties of a homogeneous material, at a given frequency ω, are ruled by two physical quantities: its permittivity ε and its permeability μ. In the range of frequencies used in optics, $\varepsilon(\omega) = \varepsilon' + \mathrm{i}\varepsilon''$ where ε' ranges over all $\mathbb{R}$ (it can be negative for metals) and $\varepsilon'' \geqslant 0$ (if we assume a time dependence in $\mathrm{e}^{-\mathrm{i}\omega t}$). In contrast almost all materials are nonmagnetic that is $\mu(\omega)$ remains very close to 1. Recently the question has been raised to know whether it is possible to produce artificial magnetism by homogenization. In particular Pendry [6] suggested to use periodic arrays of split ring resonators in 1999. This idea was followed in 2001 by Smith and Schultz [7] who obtained experimentally plausible evidence of a negative effective μ, although no clear mathematical explanation was given.

In this Note, we present a model inspired by [3] with a very simple geometry for which we can prove this effect. It consists in a periodic array of rods with high permittivity, possibly embedded in a lossless dielectric matrix.

Such a geometry was used in [5] to obtain frequency dependent effective permittivities $\varepsilon^{\mathrm{eff}}(\omega)$. The heterogeneous structure is placed in a cylinder $\mathcal{O} = \Omega \times \mathbb{R}$, where Ω is a domain of $\mathbb{R}^2$. It is composed of e_3-parallel, infinite rods whose sections are η-periodically disposed in Ω, where η is a given small parameter and $Y = [-1/2, 1/2[^2$. The rods occupy a region of space $D_\eta := \bigcup_{i \in I_\eta} \eta(i + D)$ where $D \Subset Y$ is a connected open domain with smooth boundary such that $Y^* := Y \setminus D$ is connected and $I_\eta = \{i \in \mathbb{Z}^2\colon \eta(i + D) \Subset \Omega\}$. The structure, whose relative permeability is assumed to be equal to 1, is characterized by its relative permittivity $\varepsilon_\eta = a_\eta^{-1}$ where a_η is defined on $\mathbb{R}^2$ by:

$$a_\eta(x) := \frac{\eta^2}{\varepsilon_i} \quad \text{if } x \in D_\eta, \quad a_\eta(x) := \frac{1}{\varepsilon_e} \quad \text{if } x \in \Omega \setminus D_\eta, \quad a_\eta(x) := 1 \quad \text{if } x \in \mathbb{R}^2 \setminus \Omega. \tag{1}$$

The scaling in η^2 appearing in D_η corresponds to a constant optical diameter of the fibers. We choose a time dependence in $\mathrm{e}^{-\mathrm{i}\omega t}$ and we assume that the dielectric constant ε_e of the matrix is a positive real number whereas, in the fibers, the parameter ε_i is a complex number whose imaginary part is $\geqslant 0$. The wave number k is defined by $k = \omega/c$, where c is the speed of light in vacuum. The structure is illuminated by a p-polarized incident field u^i, so that the total magnetic field $u_\eta(x_1, x_2)$ satisfies:

$$\operatorname{div}(a_\eta \nabla u_\eta) + k^2 u_\eta = 0 \quad \text{on } \mathbb{R}^2, \tag{2}$$

and the diffracted field $u_\eta^d = u_\eta - u^i$ satisfies the Sommerfeld radiation condition:

$$\frac{\partial u_\eta^d}{\partial r} - \mathrm{i}ku = \mathrm{O}\left(\frac{1}{\sqrt{kr}}\right), \quad \text{as } r \to +\infty. \tag{3}$$

Our interest is the asymptotic behavior of u_η as η tends to 0. It turns out that, due to the degeneracy of a_η in the rods, the sequence $\{u_\eta\}$ fails to be strongly relatively compact in L^2. Our aim is to characterize its two-scale limit in terms of the solution of some homogenized diffraction problem. For the notion of two-scale convergence and related topics we refer to [2].

2. The artificial magnetism

In order to state our main result, we consider the eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \cdots$ of the following Dirichlet problem

$$-\Delta\varphi = \lambda\varphi \quad \text{on } D, \qquad \varphi = 0 \quad \text{on } \partial D, \tag{4}$$

and we denote φ_n the associated normalized eigenfunctions in $H_0^1(D)$, so that $\{\varphi_n\}$ is an orthonormal basis of $L^2(D)$. We will assume that k^2 (recall that $k^2 = \omega^2/c^2$) satisfies the condition

$$k^2\varepsilon_i \notin \{\lambda_n\colon n \in J_0\} \quad \text{where } J_0 := \left\{n \geqslant 1\colon \int_D \varphi_n \neq 0\right\}. \tag{5}$$

Then, we set

$$M_\omega(y) = 1_{Y^*} + \sum_{n \in J_0} \frac{k^2\,\varepsilon_i}{\lambda_n - k^2\,\varepsilon_i}\left(\int_D \varphi_n\right)\varphi_n(y), \tag{6}$$

$$\mu^{\mathrm{eff}}(\omega) = \int_Y M_\omega \,\mathrm{d}y = 1 + \sum_{n \in J_0} \frac{k^2\varepsilon_i}{\lambda_n - k^2\varepsilon_i}\left(\int_D \varphi_n\right)^2. \tag{7}$$

Next, in a similar way as for the homogenization of Neumann problems on perforated domains, we introduce the 2×2 symmetric matrix defined by:

$$A^{\mathrm{eff}} := |Y^*| I_2 + \int_{Y^*} (\nabla w_1 | \nabla w_2)\,\mathrm{d}y, \tag{8}$$

where for $i \in \{1,2\}$, w_i is a Y-periodic solution in H^1_{loc} of the nonhomogeneous Neumann problem:

$$-\Delta w_i = 0 \quad \text{in } Y^*, \qquad \frac{\partial w_i}{\partial n} = -n_i \quad \text{on } \partial D \quad (n = (n_1, n_2) \text{ is the outer normal of} D), \tag{9}$$

and I_2 denotes the identity matrix. Finally, we set:

$$\mu(\omega, x) = \begin{cases} 1 & \text{for } x \in \mathbb{R}^2 \setminus \Omega, \\ \mu^{\mathrm{eff}}(\omega) & \text{for } x \in \Omega, \end{cases} \qquad A(x) = \begin{cases} I_2 & \text{for } x \in \mathbb{R}^2 \setminus \Omega, \\ \frac{1}{\varepsilon_e} A^{\mathrm{eff}} & \text{for } x \in \Omega. \end{cases} \tag{10}$$

The main result of this Note is the following

Theorem 2.1. *Let ω be such that* (5) *holds and let M_ω be defined by* (6)*. Then the solution u_η of* (2) *is uniformly bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and two-scale converges, as $\eta \to 0$, towards u_0 defined by*

$$u_0(x,y) := u(x) M_\omega(y) \quad \text{if } x \in \Omega, \qquad u_0(x,y) := u(x) \quad \text{if } x \in \mathbb{R}^2 \setminus \Omega,$$

where u is the unique solution in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ of the following scattering problem:

$$\begin{cases} \operatorname{div}(A(x)\nabla u) + k^2 \mu(\omega, x) u = 0 \quad \text{on } \mathbb{R}^2, \\ u - u^i \text{ satisfies } (3). \end{cases} \tag{11}$$

Remark 1. The limit equation in (11) is to be understood in the distributional sense. It can be written as a system composed of:

- the homogenized equation on Ω: $\operatorname{div}(A^{\mathrm{eff}} \nabla u) + k^2 \mu^{\mathrm{eff}}(\omega) u = 0$,
- the Helmholtz equation on $\mathbb{R}^2 \setminus \Omega$: $\Delta u + k^2 u = 0$,
- the transmission conditions on $\partial\Omega$: $(\nabla u)^+ \cdot n = (A^{\mathrm{eff}} \nabla u)^- \cdot n$, $u^+ = u^-$.

In the first equation A^{eff} stands for the inverse of the effective permittivity tensor whereas $\mu^{\mathrm{eff}}(\omega)$ represents the *frequency dependent* effective permeability.

Remark 2. It turns out that M_ω given in (6) solves the Helmholtz equation $\Delta w + k^2 \varepsilon_i w = 0$ on D with boundary condition $w = 1$ on ∂D . Actually this solution exists iff (5) holds and it is unique provided $k^2 \varepsilon_i \neq \lambda_n$ for all n (which is always true when ε_i has a positive imaginary part). When condition (5) is violated, it is possible to prove that $u_\eta \to u$ in L^2_{loc} where u vanishes on Ω and solves the Dirichlet exterior problem for Helmholtz equation on $\mathbb{R}^2 \setminus \Omega$.

Remark 3. We emphasize that, due to the y-dependence of the two-scale limit $u_0(x,y)$, the convergence of $\{u_\eta\}$ cannot be strong in $L^2(\Omega)$ unless $u \equiv 0$. The weak limit of u_η in $L^2(\Omega)$ is given by $\tilde{u}(x) = u(x)\mu^{\mathrm{eff}}(\omega)$ (which differs from $u(x)$). In contrast, away from the structure, we have that $u_\eta \to u$ strongly in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$.

To conclude this section, we mention some important physical issues linked to the behavior of μ^{eff} as a function of the frequency ω. The key point is that electromagnetic waves ruled by the homogenized equation (11) cannot propagate and are damped inside Ω when $\mu^{\mathrm{eff}} < 0$ (as $\varepsilon_e > 0$, A^{eff} is a definite positive matrix). According to (7), this damping happens on a countable union of frequency intervals. Such forbidden bands are photonic band gaps. It is worth noting that this effect is due to internal resonances and not to energy dissipation. In particular we may take ε_i to be a positive real number. The behavior of $\mu^{\mathrm{eff}}(\omega)$ in this case is quite simple: let $\omega_l = c\sqrt{\lambda_{n_l}/\varepsilon_i}$ where $\{\lambda_{n_l}: l \in \mathbb{N}\}$ are the $\{\lambda_n,\ n \in J_0\}$ sorted in ascending order. Then $\mu^{\mathrm{eff}}(\omega)$ decreases continuously from $+\infty$ to $-\infty$ on each interval (ω_l, ω_{l+1}) passing through zero at a unique point $\overline{\omega_l}$. The interval $[\overline{\omega_l}, \omega_{l+1}]$ forms a photonic band gap.

3. Sketch of proofs

Step 1. We first assume that $\{u_\eta\}$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Then, as u_η satisfies Helmholtz equation $(\Delta+k^2)u_\eta = 0$ on $\mathbb{R}^2\setminus\Omega$, it is relatively compact in $C^\infty_{\mathrm{loc}}(\mathbb{R}^3\setminus\Omega)$. Multiplying Eq. (2) by $\overline{u_\eta}$ and integrating by parts on a ball $B_R := \{|x|<R\}$ where R is fixed so large that $\Omega\subset B_R$, we derive that

$$\sup\left\{\int_{B_R} a_\eta(x)\,|\nabla u_\eta|^2\right\} < +\infty. \tag{12}$$

From (12), we deduce that the sequence $(\chi_\eta, \eta\nabla u_\eta)$ where $\chi_\eta := \nabla u_\eta 1_{\mathbb{R}^2\setminus D_\eta}$ is bounded in $L^2(B_R)$. Then, possibly after extracting subsequences, we may assume that

$$u_\eta \rightharpoonup\rightharpoonup u_0(x,y),\quad \chi_\eta \rightharpoonup\rightharpoonup \chi_0(x,y),\quad \eta\,\nabla u_\eta \rightharpoonup\rightharpoonup D_0(x,y), \tag{13}$$

where $u_0(x,y)$, $\chi_0(x,y)$, $D_0(x,y)$ are functions in $L^2(B_R\times Y)$ (extended by periodicity to all $y\in\mathbb{R}^2$). In what follows, $H^1_\sharp$ denotes the space of Y-periodic functions in H^1_{loc}.

Lemma 3.1. (i) *For a.e. $x\in\Omega$, the function $u_0(x,\cdot)$ belongs to $H^1_\sharp$, is constant on Y^* and satisfies the equation $\Delta_y u_0 + k^2\,\varepsilon_i u_0 = 0$ on D. For $x\in B_R\setminus\Omega$, $u_0(x,\cdot)$ is constant over all Y.*

(ii) *For a.e. $x\in\Omega$, the function $\chi_0(x,\cdot)$ is divergence free.*

Proof. (i) Applying (13) to test functions $\varphi(x,y)$ such that $\varphi(x,\cdot)$ is compactly supported in D or in Y^*, we easily derive that, for almost all $x\in\Omega$, $\nabla_y u_0(x,\cdot)$ belongs to L^2_{loc} and satisfies

$$\nabla_y u_0(x,\cdot) = D_0(x,\cdot)\quad \text{on } D,\qquad \nabla_y u_0(x,\cdot)=0\quad\text{on } Y^*. \tag{14}$$

By the connectedness property of Y^*, the second equality implies that $u_0(x,\cdot)$ as an element of $H^1_\sharp$, is constant on Y^*. For $x\in B_R\setminus\Omega$, $u_0(x,\cdot)$ is obviously constant since, by (12), $\{u_\eta\}$ is bounded in $H^1(B_R\setminus\Omega)$.

Now multiplying (2) by a test function $\varphi_\eta := \varphi(x,x/\eta)$ compactly supported in Ω and integrating by parts, we obtain

$$\int_\Omega a_\eta\nabla u_\eta\cdot\left[\left(\nabla_x\varphi + \frac{1}{\eta}\nabla_y\varphi\right)\right]\left(x,\frac{x}{\eta}\right)\mathrm{d}x = k^2\int_\Omega u_\eta\varphi\left(x,\frac{x}{\eta}\right)\mathrm{d}x. \tag{15}$$

First we take φ compactly supported in $\Omega\times D$. Then passing to the limit in (15) and localizing in x, we derive that $\mathrm{div}_y\, D_0(x,\cdot) + k^2u_0(x_0,\cdot)=0$. In view of (14), we deduce the Helmholtz equation satisfied by u_0 on D. Taking now φ compactly supported in $\Omega\times Y^*$ and multiplying (15) by η before passing to the limit, we derive that $\mathrm{div}_y\,\chi_0 = 0$. □

Let us denote by $u(x)$ the constant value associated with $u_0(x,\cdot)$. We have

Lemma 3.2. *u belongs to $H^1(B_R)$ and there exists an element $u_1\in L^2(\Omega; H^1_\sharp(Y))$ such that*

$$\chi_0(x,y) = \big(\nabla u(x) + \nabla_y u_1(x,y)\big)1_{Y^*}(y)\quad \textit{if } x\in\Omega,\qquad \chi_0(x,y)=\nabla u(x)\quad \textit{if } x\in B_R\setminus\Omega. \tag{16}$$

Proof. By the uniform extension lemma applied to the perforated set $\Omega\setminus D_\eta$ (see [1]), there exists a bounded sequence $\{\tilde u_\eta\}$ in $H^1(\Omega)$ such that $\tilde u_\eta = u_\eta$ on $B_R\setminus D_\eta$. Clearly u defined above is the unique possible cluster point of $\{\tilde u_\eta\}$ in $L^2(B)$. Therefore u belongs to $H^1(B)$ and $\chi_0(x,y)$ agrees with the two-scale limit of $\{\nabla\tilde u_\eta\}$ on $\Omega\times Y^*$, whereas it vanishes on $\Omega\times D$. Thus (see [2]), there exits an element u_1 of $L^2(\Omega; H^1_\sharp(Y))$ such that (16) holds on Ω. On $B_R\setminus\Omega$, the corrector u_1 vanishes since (see below) the convergence $u_\eta\to u$ holds in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2\setminus\overline{\Omega})$. □

Lemma 3.3. *For a.e. $x \in \Omega$, we have*:

(i) $u_0(x,\cdot) = u(x)M_\omega(\cdot)$ *if* (5) *holds and* $u_0(x,\cdot) \equiv 0$ *otherwise.*
(ii) $\int_Y \chi_0(x,y)\,\mathrm{d}y = A^{\mathrm{eff}}\nabla u(x)$.

Proof. (i) We write $u_0(x,y) = u(x)(1+w(y))$ where, by Lemma 3.1, $w \in H_0^1(D)$ satisfies $\Delta w + k^2 w = k^2$ on D. Then, for $k^2\varepsilon_i \neq \lambda_n$, we express uniquely w in terms of its components on the basis $\{\varphi_n\}$. This representation formula can be extended to all k^2 satisfying (5) by using a continuity argument. When (5) is violated, the equation in w has no solution and we are led to $u(x) = u_0(x,y) = 0$.

(ii) By Lemma 3.1, $\chi_0(x,\cdot)$ given by (16) is divergence free. Thus $u_1(x,\cdot)$ solves a Neumann problem on Y^* whose solution agrees, up to an additive constant, with $u_1(x,y) = (\partial_1 u)(x)w_1(y) + (\partial_2 u)(x)w_2(y)$ where w_1, w_2 are defined in (9). The conclusion follows by substituting the latter expression in (16) and by integrating in y. □

In order to complete step 1, we simply observe that, according to (1), (13), we have

$$a_\eta \nabla u_\eta \rightharpoonup \sigma_0(x,y) := \begin{cases} \frac{1}{\varepsilon_i}\chi_0(x,y) & \text{if } x \in \Omega, \\ \frac{1}{\varepsilon_i}\nabla u(x) & \text{if } x \in B_R \setminus \Omega. \end{cases}$$

Therefore, in view of Lemma 3.1 and of the assertion (ii) of Lemma 3.3, taking into account (10) and (7), we infer that the weak limits in $L^2(B_R)$ of $\{u_\eta\}$ and $\{a_\eta \nabla u_\eta\}$ are given respectively by

$$\tilde{u}(x) = \int_Y u_0(x,y)\,\mathrm{d}y = \mu(x)u(x), \qquad \sigma(x) := \int_Y \sigma_0(x,y)\,\mathrm{d}y = A(x)\nabla u(x). \tag{17}$$

The limit equation in (11) follows from (17) by passing to the limit in (2). Eventually, the radiation condition (3), which can be written in the Kirchhoff integral form (see for example [4]), is preserved since by the hypo-ellipticity of Helmholtz equation, the convergence $u_\eta \to u$ holds in fact in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$.

Step 2. We prove that $\{u_\eta\}$ is bounded in L^2_{loc} by contradiction assuming that $\|u_\eta\|_{L^2(B_R)} = t_\eta \to \infty$. Dividing the incident wave u^i by t_η, we apply the conclusions of step 1 to the normalized solutions $\overline{u_\eta} := u_\eta/t_\eta$ to find a limit which, by the uniqueness of the solution of system (11), is identically zero. Then we conclude to a contradiction by proving that $\overline{u_\eta} \to 0$ strongly in $L^2(B_R)$. □

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References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Anal. 18 (5) (1992) 481–496.
[2] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (1992) 1482–1518.
[3] S. O'Brien, J.B. Pendry, Photonic band-gaps effects and magnetic activity in dielectric composites, J. Phys. Condens. Mat. 14 (2002) 4035–4044.
[4] M. Cessenat, Mathematical Methods in Electromagnetism. Linear Theory and Applications, Ser. Adv. Math. Appl. Sci., vol. 41, World Scientific, 1996.
[5] D. Felbacq, G. Bouchitté, Homogenization of a set of parallel fibers, Waves Random Media 7 (1997) 245–256.
[6] J.B. Pendry, A.J. Holden, D.J. Robbins, W.J. Stewart, Magnetism from conductors, and enhanced non-linear phenomena, IEEE T. Microw. Theory 47 (1999) 2075–2084.
[7] R.A. Shelby, D.R. Smith, S.C. Nemat-Nasser, S. Schultz, Microwave transmission through a two-dimensional, isotropic, left-handed metamaterial, Appl. Phys. Lett. 78 (2001) 489–491.