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Mathematical Problems in Mechanics/Partial Differential Equations

Homoclinic solutions of reversible systems possessing an essential spectrum

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Abstract

In this Note we consider bifurcations of a class of infinite dimensional reversible dynamical systems. These systems possess a family of equilibrium solutions near the origin. We also assume that the linearized operator at the origin L_ε has an essential spectrum filling the entire real line, in addition to a simple eigenvalue at 0. Moreover, for parameter values $\varepsilon < 0$ there is a pair of imaginary eigenvalues which meet in 0 for $\varepsilon = 0$, and which disappear for $\varepsilon > 0$. We give assumptions on L_ε and on the non-linear term which describe this situation. These assumptions are sufficient to prove the existence of a family of solutions homoclinic to the equilibrium solutions near the origin. The result of this Note applies when we look for solitary waves in superposed layers of perfect fluids, the bottom one being infinitely deep. ***To cite this article: M. Barrandon, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Solutions homoclines de systèmes réversibles en présence d'un spectre essentiel. On étudie les bifurcations d'une classe de systèmes dynamiques réversibles de dimension infinie. Ces systèmes possèdent une famille de solutions stationnaires près de l'origine. On suppose que l'opérateur linéarisé à l'origine L_ε a un spectre essentiel sur l'axe réel et une valeur propre simple en 0. Une paire de valeurs propres imaginaires pour les valeurs du paramètre $\varepsilon < 0$ se rencontrent à l'origine pour $\varepsilon = 0$ et disparaissent pour $\varepsilon > 0$. On donne ici des hypothèses sur L_ε et sur le terme non linéaire qui précisent la situation. Avec ces hypothèses on montre l'existence d'une famille de solutions homoclines aux solutions d'équilibre près de l'origine. Ce résultat s'applique à la recherche d'ondes solitaires dans des couches superposées de fluides parfaits, la couche inférieure étant de profondeur infinie. ***Pour citer cet article : M. Barrandon, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

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On étudie les bifurcations d'une classe de systèmes dynamiques réversibles de dimension infinie dans un espace de Banach $\mathbb{H}$

$$\frac{\mathrm{d}U}{\mathrm{d}x} = L_\varepsilon U + N_\varepsilon(U), \quad U(x) \in \mathbb{D} \subset \mathbb{H}, \tag{1}$$

ce système possède une solution d'équilibre à l'origine et l'opérateur linéarisé à l'origine L_ε a un spectre essentiel sur l'axe réel et une valeur propre simple en 0, correspondant à l'existence d'une famille à un paramètre d'équilibres réversibles. On suppose également que L_ε a une paire de valeurs propres imaginaires pour $\varepsilon < 0$ qui se rencontrent en 0 pour $\varepsilon = 0$ et disparaissent dans le spectre essentiel pour $\varepsilon > 0$. On note S la symétrie de réversibilité (S anticommute avec L_ε et N_ε). On suppose finalement l'existence d'un changement d'échelle en variables et en coordonnées qui dilate d'un facteur $1/\varepsilon$ le spectre de L_ε, donnant un nouvel opérateur linéaire $\mathcal{L}_\varepsilon$ pour $\varepsilon > 0$, et un nouvel opérateur non linéaire $\mathcal{N}_\varepsilon$. L'Éq. (1) devient

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \mathcal{L}_\varepsilon U + \mathcal{N}_\varepsilon(U), \quad U(x) \in \mathbb{D}. \tag{2}$$

On donne des hypothèses sur $\mathcal{L}_\varepsilon$ et $\mathcal{N}_\varepsilon$ sous lesquelles on montre l'existence d'une famille à un paramètre de solutions stationnaires de (2) et l'existence d'une famille de solutions homoclines à ces solutions stationnaires. La théorie des vagues fournit des exemples de systèmes dynamiques qui vérifient ces hypothèses (voir [5]). En effet, si on considère deux couches de fluides parfaits superposées, la couche supérieure étant bornée par une surface rigide (voir [1,8]) ou possédant une surface libre avec une forte tension de surface, la couche inférieure étant de profondeur infinie, alors la recherche d'ondes progressives conduit à un système dynamique réversible ayant les mêmes propriétés que (2) (voir [3]).

La principale hypothèse (voir H1 dans la Section 3) décrit la résolvante de $\mathcal{L}_\varepsilon$ pour k réel non nul et $\varepsilon|k| < \delta$. Soit $V \in \mathbb{H}$ alors

$$(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1} V = \frac{\xi^*_{\varepsilon,k}(V)}{\mathrm{i}k\varepsilon\Delta}\xi_0 + \frac{\eta^*_{\varepsilon,k}(V)}{\Delta}\theta_k + \varepsilon S_{\varepsilon,k}(V), \tag{3}$$

où ξ_0 est un vecteur propre (symétrique) associé à la valeur propre 0, le vecteur θ_k est dans $\mathbb{D}$ pour $k \neq 0$, mais n'est pas continu en $k = 0$. La fonction Δ vérifie $\Delta(\varepsilon, k) = 1 + a|k| + \mathrm{O}(\varepsilon k^2)$ avec $a > 0$. Pour $k \neq 0$, les formes linéaires $\xi^*_{\varepsilon,k}$, $\eta^*_{\varepsilon,k}$ sont dans $\mathbb{H}^*$ et $S_{\varepsilon,k} \in \mathcal{L}(\mathbb{H}, \mathbb{D})$. La nature de la singularité en $k = 0$ fait intervenir les espaces $\widetilde{\mathbb{H}}$ et $\widehat{\mathbb{D}}$ comme indiqué dans H1. On suppose également l'identité (7) où $\xi^*_\varepsilon = \xi^*_{\varepsilon,0}$. Cette hypothèse n'est pas restrictive car elle est automatiquement vérifiée à un facteur près, on obtient alors (7) pourvu que ce facteur soit non nul. On donne ensuite des précisions sur la régularité des opérateurs introduits dans H1 avec les hypothèses H2 et H3. En particulier, on définit dans H3 l'opérateur $u \mapsto \mathcal{T}(u) = \mathcal{F}^{-1}(-\mathrm{i}k\hat{u}\theta_k)$ et on suppose qu'il est régulier dans des espaces de fonctions à décroissance polynomiale à l'infini. L'hypothèse H4 décrit la résolvante pour des grandes valeurs de $\varepsilon|k|$, on suppose que celle-ci se comporte en $1/k$.

On précise dans H5 la régularité du terme non linéaire et on suppose que $\mathcal{N}_\varepsilon(u\xi_0) = 0$ pour tout u réel, ce qui implique que $U(x) = u\xi_0$ est une solution stationnaire de (2) pour tout $u \in \mathbb{R}$. On suppose finalement la continuité en $k = 0$ des deux termes suivants : $\xi^*_\varepsilon(\partial_u \mathcal{N}_\varepsilon(u\xi_0 + Y)_{|u=0}.\theta_k)$ et $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$. En particulier la valeur en $k = 0$ de $\xi^*_\varepsilon(\partial_u \mathcal{N}_\varepsilon(u\xi_0 + Y)_{|u=0}.\theta_k)$ est notée $2c_0$ et est supposée non nulle (voir H6 et H7).

On obtient alors le résultat suivant (voir la Section 2 pour les définitions des espaces et de la fonction u_h) qui montre l'existence d'une famille de solutions réversibles homoclines aux solutions stationnaires $U(x) = u\xi_0$ pour $|u|$ petit. On remarque que la décroissance de ces solutions quand $x \to \pm\infty$ est polynomiale.

Théorème 0.1. *On suppose que* $\mathcal{L}_\varepsilon$ *satisfait* H1–H3 *et* H4 *et que* $\mathcal{N}_\varepsilon$ *satisfait* H5, H6 *et* H7. *Alors pour tout* $0 < \alpha < 1$, *il existe* $\varepsilon_0 > 0$ *et* $u_0 > 0$ *tels que, pour tout* $0 < \varepsilon < \varepsilon_0$ *et pour tout* $0 \leqslant |u| < u_0\varepsilon$ (*si* $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$ *est indépendant de k alors* $0 \leqslant |u| < u_0$), *l'Éq.* (2) *possède une solution réversible* $U_{\varepsilon,u}$ *qui vérifie*

$$U_{\varepsilon,u}(x) = \big(u + u_h(x)\big)\xi_0 + \widetilde{U}_{\varepsilon,u}(x),$$

où u_h est l'homocline de Benjamin–Ono et où le terme $\widetilde{U}_{\varepsilon,u} \in B_2^{1,\alpha}(\mathbb{R})\xi_0 \oplus \mathcal{B}^\alpha(\widehat{\mathbb{D}})$ (donc décroit comme $1/x^2$ à l'infini) est d'ordre $\mathrm{O}(\varepsilon + |u|\varepsilon^{-1})$, ou $\mathrm{O}(\varepsilon + |u|)$ si $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$ est indépendant de k.

1. Introduction

In this work we consider bifurcations of a class of infinite dimensional reversible dynamical systems of the following form in a Banach space $\mathbb{H}$

$$\frac{\mathrm{d}U}{\mathrm{d}x} = L_\varepsilon U + N_\varepsilon(U), \quad U(x) \in \mathbb{D} \subset \mathbb{H}, \tag{4}$$

possessing an equilibrium solution at the origin, such that the linearized operator at the origin L_ε has an essential spectrum filling the entire real line, in addition to a simple eigenvalue at 0, corresponding to the existence of a one parameter family of reversible equilibra. We also assume that for parameter values $\varepsilon < 0$ there is a pair of imaginary eigenvalues which meet in 0 for $\varepsilon = 0$ and which disappear for $\varepsilon > 0$. We denote by S the symmetry of reversibility (S anti-commutes with L_ε and N_ε). We finally assume that for $\varepsilon > 0$, there is a rescaling in coordinates and variables which dilates the original spectrum by a factor $1/\varepsilon$ and gives the new linear operator $\mathcal{L}_\varepsilon$, the new non-linear term is denoted by $\mathcal{N}_\varepsilon$, and (4) now reads

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \mathcal{L}_\varepsilon U + \mathcal{N}_\varepsilon(U), \quad U(x) \in \mathbb{D}. \tag{5}$$

In this Note we give sufficient assumptions on $\mathcal{L}_\varepsilon$ and $\mathcal{N}_\varepsilon$ to prove the existence of a one parameter family of stationary solutions, and the existence of a family of reversible solutions homoclinic to these stationary solutions. Such dynamical systems can be derived in the theory of water-waves [3,5]. Indeed, looking for traveling waves in two superposed layers of perfect fluids, the upper one being bounded by a rigid top [1,8] or having a free surface with a sufficiently large surface tension, the bottom one being infinitely deep, leads to a dynamical system having the same properties as (5) (see [3]).

2. Definitions

Before giving the assumptions on the resolvent operator we must introduce some spaces. We introduce the Banach spaces $\widetilde{\mathbb{H}} \subset \mathbb{H}$ and $\widetilde{\mathbb{D}} \subset \mathbb{D}$ where $\widetilde{\mathbb{H}}$ is dense in $\mathbb{H}$ and $\widetilde{\mathbb{D}}$ is dense in $\mathbb{D}$ and such that $\mathcal{L}_\varepsilon : \mathbb{D} \mapsto \mathbb{H}$ and $\widetilde{\mathbb{D}} \mapsto \widetilde{\mathbb{H}}$. Actually, $\mathcal{L}_\varepsilon$ turns out to be Fredholm in $\widetilde{\mathbb{H}}$. Introducing $\widetilde{\mathbb{H}}$ is necessary for describing the smoothness in k near 0 of some important k dependent operators occurring in the detailed form of the resolvent $(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1}$ explicitly given in the forthcoming hypothesis H1. Notice that we chose to work in a space $\mathbb{H}$ where $\mathcal{L}_\varepsilon$ is not Fredholm. This is due to the fact that non-trivial solutions U decaying as $|x| \to \infty$ does not lead to solutions such that $U(x)$ lies in $\widetilde{\mathbb{H}}$. In the theory of water-waves, $\widetilde{\mathbb{H}}$ corresponds to a space in which the functions describing the infinitely deep layer of fluid have an exponential decay rate at infinity in the vertical variable, whereas we look for solutions in $\mathbb{H}$, with an polynomial decay rate at infinity (see [3,6]).

We also need a Banach space $\widehat{\mathbb{D}} : \mathbb{D} \subset \widehat{\mathbb{D}} \subset \mathbb{H}$, which is used in the description of the regularity of the resolvent operator $(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1}$. This operator, considered in $\mathcal{L}(\mathbb{H}, \mathbb{D})$ is a smooth function of k in $\mathbb{R}\backslash\{0\}$ and the singularity in $k = 0$ is controlled in $\mathcal{L}(\widetilde{\mathbb{H}}, \widehat{\mathbb{D}})$ (see H1). At last, $\widetilde{\mathbb{H}}$ and $\widehat{\mathbb{D}}$ must be chosen so that $\mathcal{N}_\varepsilon : \widehat{\mathbb{D}} \to \widetilde{\mathbb{H}}$.

To describe the singularity in $k = 0$ of the resolvent $(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1}$, we introduce for a Banach space $\mathbb{E}$ the space of functions

$$\mathcal{C}^p_{\lim}(\mathbb{R}, \mathbb{E}) = \left\{ f : \mathbb{R} \to \mathbb{E};\ f \in \mathcal{C}^p\left(\mathbb{R}\backslash\{0\}\right),\ f \text{ continuous in 0 and } f^{(n)} \text{ has limits in } 0^\pm\ \forall n \leqslant p \right\}.$$

We also define the following spaces depending on $\varepsilon : \mathcal{C}^p_{\lim,\varepsilon}(\mathbb{R}, \mathbb{E}) = \mathcal{C}^p_{\lim}(\mathbb{R}, \mathbb{E}) \cap \{\|f^{(n)}\|_{\mathbb{E}} \leqslant c\varepsilon^n \text{ for } n \leqslant p\}$.

We now introduce the Banach spaces which describe the dependence in x of the solutions of (5). Let $0 < \alpha < 1$, then we define the Hölder space $B^\alpha_p(\mathbb{E}) = \{f \in \mathcal{C}^\alpha(\mathbb{R}, \mathbb{E});\ \|f\|^\alpha_{p,\mathbb{E}} < \infty\}$, where

$$\|f\|^{\alpha}_{p,\mathbb{E}} = \sup_{x\in\mathbb{R}}\bigl(1+|x|^p\bigr)\bigl\|f(x)\bigr\|_{\mathbb{E}} + \sup_{x\in\mathbb{R},\ 0<|\delta|\leqslant 1}\bigl(1+|x|^p\bigr)\frac{\|f(x+\delta)-f(x)\|_{\mathbb{E}}}{|\delta|^{\alpha}}.$$

We also need $B_2^{1,\alpha}(\mathbb{R})$ defined by $B_2^{1,\alpha}(\mathbb{R}) = \{f \in B_2^{\alpha}(\mathbb{R});\ \frac{\mathrm{d}f}{\mathrm{d}x} \in B_2^{\alpha}(\mathbb{R})\}$, with the corresponding norm denoted by $\|f\|^{1,\alpha}_{2,\mathbb{R}}$.

In the following, the Hilbert transform $\mathcal{H}(f)$ of a function f is defined by $\mathcal{H}(f)(x) = p.v.\int_{\mathbb{R}} \frac{f(s)}{x-s}\,\mathrm{d}s$.

3. Assumptions on the linear term

We now state the main assumption on the linear operator. This assumption gives the structure of the resolvent operator of $\mathcal{L}_\varepsilon$ near the origin and describes the spectrum of $\mathcal{L}_\varepsilon$ and its singularity in $k=0$.

H1 (*resolvent for small $\varepsilon|k|$*). *There exists* $\delta > 0$ such that for $k \in \mathbb{R}\backslash\{0\}$, $\varepsilon|k| \leqslant \delta$ and $V \in \mathbb{H}$

$$(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1}V = \frac{\xi^*_{\varepsilon,k}(V)}{\mathrm{i}k\varepsilon\Delta}\xi_0 + \frac{\eta^*_{\varepsilon,k}(V)}{\Delta}\theta_k + \varepsilon S_{\varepsilon,k}(V), \tag{6}$$

with the following properties

(i) the map $k \mapsto \Delta \in \mathcal{C}^3_{\lim}(\mathbb{R},\mathbb{R})$ is even and satisfies $\Delta = 1 + a|k| + \mathrm{O}(\varepsilon k^2)$ with $a > 0$,
(ii) the vector $\xi_0 \in \mathbb{D}$ is symmetric ($S\xi_0 = \xi_0$) and satisfies $\mathcal{L}_\varepsilon \xi_0 = 0$ and $\ker \mathcal{L}_\varepsilon$ has dimension 1,
(iii) the vector $\theta_k \in \mathbb{D}$ for $k \neq 0$, the map $k \mapsto \theta_k$ is $\mathcal{C}^0(\mathbb{R}\backslash\{0\},\mathbb{D})$, $k\theta_k$ is bounded in $\mathbb{D}$ and $S\theta_k = -\theta_{-k}$,
(iv) for $k \neq 0$, $\xi^*_{\varepsilon,k}$ and $\eta^*_{\varepsilon,k} \in \mathbb{H}^*$ and the map $k \mapsto k\xi^*_{\varepsilon,k}$ is $\mathcal{C}^0(\mathbb{R},\mathbb{H}^*)$. Moreover

$$\begin{cases} \xi^*_{\varepsilon,k} = \xi^*_\varepsilon + \varepsilon|k|\chi^*_\varepsilon + \zeta^*_{\varepsilon,k}, \\ \eta^*_{\varepsilon,k} = \eta^*_\varepsilon + \beta^*_{\varepsilon,k} \end{cases}$$

with $\xi^*_\varepsilon \in \mathbb{H}^*$, η^*_ε, $\chi^*_\varepsilon \in \widetilde{\mathbb{H}}^*$, $\zeta^*_{\varepsilon,0} = \beta^*_{\varepsilon,0} = 0$, $k \mapsto \zeta^*_{\varepsilon,k}$ and $k \mapsto \beta^*_{\varepsilon,k} \in \mathcal{C}^1(\mathbb{R},\widetilde{\mathbb{H}}^*) \cap \mathcal{C}^3_{\lim,\varepsilon}(\mathbb{R},\widetilde{\mathbb{H}}^*)$ and

$$\xi^*_\varepsilon(\theta_k) = \mathrm{i}\,\mathrm{sgn}(k), \tag{7}$$

(v) $S_{\varepsilon,k} \in \mathcal{L}(\mathbb{H},\mathbb{D})$ for $k \neq 0$ and $k \mapsto S_{\varepsilon,k} \in \mathcal{C}^3_{\lim,\varepsilon}(\mathbb{R},\mathcal{L}(\widetilde{\mathbb{H}},\widehat{\mathbb{D}}))$,
(vi) there exists $p^*_0 \in \mathbb{H}^*$ such that $p^*_0(SV) = p^*_0(V)$, $p^*_0(\xi_0) = 1$, $p^*_0(\theta_k) = p^*_0(S_{\varepsilon,k}(V)) = 0$ and $p^*_0(\mathcal{L}_\varepsilon Y) : \mathbb{D} \to \mathbb{R}$ can be extended in a continuous map: $\widehat{\mathbb{D}} \to \mathbb{R}$.

Note that (7) is not a strong assumption since it is satisfied up to a multiplicative factor, then (7) holds as soon as this factor is not zero.

To precise the behavior of θ_k in the neighborhood of 0 we need to introduce a new Banach space. This space is also important in the construction of the solutions of the equation because this construction strongly depends on the structure of the vector θ_k. We introduce a Banach space $\mathcal{B}^\alpha(\widehat{\mathbb{D}})$ such that $\mathcal{B}^\alpha(\widehat{\mathbb{D}}) \supset B_2^\alpha(\widehat{\mathbb{D}})$, with the associated norm $\|\|_{\mathcal{B}}$, and its main properties are indicated in the following hypotheses H2 and H3. We assume that for $U \in \mathcal{B}^\alpha(\widehat{\mathbb{D}})$ and any $x \in \mathbb{R}$ then $U(x) \in \widehat{\mathbb{D}}$. Actually this space is a little larger than $B_2^\alpha(\widehat{\mathbb{D}})$ and this freedom is needed in our water waves examples due to the link between the decay in x as $x \to \pm\infty$ and the decay in the vertical coordinate in the infinitely deep fluid layer. One of the main assumption on $\mathcal{B}^\alpha(\widehat{\mathbb{D}})$ is the following

H2. The linear forms ξ^*_ε and $p^*_0\mathcal{L}_\varepsilon$ are continuous from $\mathcal{B}^\alpha(\widehat{\mathbb{D}})$ to $B_2^\alpha(\mathbb{R})$.

Next hypothesis explains the link between $\mathcal{B}^\alpha(\widehat{\mathbb{D}})$ and the vector θ_k.

H3 (*"regularity" of* $\mathrm{i}k\theta_k$). For any $u \in B_2^{1,\alpha}(\mathbb{R})$ we define the map

$$u \mapsto \mathcal{T}(u) = \mathcal{F}^{-1}(-\mathrm{i}k\hat{u}\theta_k), \tag{8}$$

and we assume that $\mathcal{T} \in \mathcal{L}(B_2^{1,\alpha}(\mathbb{R}), \mathcal{B}^\alpha(\widehat{\mathbb{D}}))$.

Remark 1. A consequence of hypothesis H1 and H2 is $\xi_\varepsilon^*(\mathcal{T}(u)) = \mathcal{H}(\frac{\mathrm{d}u}{\mathrm{d}x})$.

We finally give the behavior of the resolvent operator when $\varepsilon|k| \geqslant \delta/2$.

H4 (*resolvent operator for large* $\varepsilon|k|$). Let $V \in \mathbb{H}$, then

$$(\mathrm{i}k - \mathcal{L}_\varepsilon)^{-1} V = G(\varepsilon, k)(V), \tag{9}$$

where $k \mapsto G(\varepsilon, k)$ is continuously differentiable in $\mathcal{L}(\mathbb{H}, \mathbb{D})$ for $\varepsilon|k| \geqslant \delta/2$ with the following estimates in $\mathcal{L}(\mathbb{H})$ and in $\mathcal{L}(\widetilde{\mathbb{H}}, \widehat{\mathbb{D}})$ ($\varepsilon|k| \geqslant \delta/2$), for $n = 0, 1, 2, 3$

$$\left\|\partial_k^n G(\varepsilon, k)\right\|_{\mathcal{L}(\mathbb{H})} \leqslant c/|k|^{n+1}, \tag{10}$$

$$\left\|\partial_k^n G(\varepsilon, k)\right\|_{\mathcal{L}(\widetilde{\mathbb{H}}, \widehat{\mathbb{D}})} \leqslant c\varepsilon/|k|^{n}. \tag{11}$$

The above assumptions mean that the resolvent operator behaves like $1/k$ in $\mathcal{L}(\mathbb{H})$ and that we obtain a bound of the norm in $\widehat{\mathbb{D}}$ in multiplying by εk.

4. Assumptions on the non-linear term

We give in this section the assumptions on the non-linear term.

H5 (*regularity of the non-linear term*). We assume that

(i) $\mathcal{N}_\varepsilon(u\xi_0) = 0$, $\forall u \in \mathbb{R}$,
(ii) $\mathcal{N}_\varepsilon \in \mathcal{C}^k(\widehat{\mathbb{D}}, \widetilde{\mathbb{H}})$, and more precisely

$$\mathcal{N}_\varepsilon \in \mathcal{C}^k\big(\mathcal{B}^\alpha(\widehat{\mathbb{D}}), B_3^\alpha(\widetilde{\mathbb{H}})\big),$$

with $k \geqslant 3$ and $\mathcal{N}_\varepsilon(U) = N_2(U, U) + \mathrm{O}(\varepsilon)$ for $\|U\|_{\mathcal{B}} \leqslant M$ and where N_2 is quadratic. We finally assume that $D\mathcal{N}_\varepsilon(0) = 0$ and

$$D^m\mathcal{N}_\varepsilon(0) = \mathrm{O}(\varepsilon^{m-2}), \quad m = 2, 3.$$

Note that H5(i) implies the existence of the stationary solutions $U(x) = u\xi_0$. We now search reversible solutions of (5) of the form $U = (u + w)\xi_0 + \varepsilon Y$, $p_0^*(Y) = 0$ (since U is reversible we deduce that w is even and that $SY(x) = Y(-x)$). Then we must compute the non-linear term for such a vector U. From Hypothesis H5 there is a smooth function R_ε such that

$$\mathcal{N}_\varepsilon(u\xi_0 + \varepsilon Y) = \varepsilon R_\varepsilon(u, Y). \tag{12}$$

From H5 the map $R_\varepsilon : B_2^\alpha(\mathbb{R}) \times \mathcal{B}^\alpha(\widehat{\mathbb{D}}) \to B_3^\alpha(\widetilde{\mathbb{H}})$ is $\mathcal{C}^k$, $k \geqslant 3$ and satisfies

$$R_\varepsilon(u, Y) = uD_\varepsilon Y + \widetilde{R}_\varepsilon(u, Y),$$

with $D_\varepsilon \in \mathcal{L}(\widehat{\mathbb{D}}, \widetilde{\mathbb{H}})$ and $(u, Y) \mapsto uD_\varepsilon Y \in \mathcal{L}(B_2^\alpha(\mathbb{R}) \times \mathcal{B}^\alpha(\widehat{\mathbb{D}}), B_3^\alpha(\widetilde{\mathbb{H}}))$. Moreover the following estimate holds for $\|u\|_{2,\mathbb{R}}^{1,\alpha} + \|Y\|_{\mathcal{B}} \leqslant M$

$$\|\widetilde{R}_\varepsilon\|_{3,\widetilde{\mathbb{H}}}^\alpha \leqslant \varepsilon\|Y\|_{\mathcal{B}}\big(\|u\|_{2,\mathbb{R}}^{1,\alpha} + \|Y\|_{\mathcal{B}}\big).$$

Notice that the main term of R_ε is $uD_\varepsilon Y$. The linear map D_ε introduced here plays an important role. Actually a very important coefficient turns out to be $\frac{1}{2}a\xi_\varepsilon^*(D_\varepsilon\theta_k)$, which is the coefficient of the lowest order non-linear term in the equation for the Fourier transform $\hat{w}$. In order to obtain a local non-linear term in the equation for w we need to assume the following.

H6. Assume that the map $k \mapsto \xi_\varepsilon^*(D_\varepsilon\theta_k)$ is continuous in $k=0$ and

$$\xi_\varepsilon^*(D_\varepsilon\theta_k) = 2c_0 + \varepsilon\gamma_\varepsilon(k), \quad c_0 \neq 0, \tag{13}$$

where $k \mapsto \gamma_\varepsilon(k) \in \mathcal{C}^2_{\mathrm{lim},\varepsilon}(\mathbb{R},\mathbb{R})$ for $\varepsilon|k| < \delta$.

We add an assumption on the continuity in $k=0$ in $\widetilde{\mathbb{H}}$ of the coefficient $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$, whereas θ_k is not continuous in $\mathbb{D}$ in $k=0$.

H7. We assume that $k \mapsto D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$ is in $\mathcal{C}^3_{\mathrm{lim}}(\mathbb{R},\widetilde{\mathbb{H}})$ for $|u| < u_0$ and $\varepsilon|k| < \delta$.

5. Result

Before giving the result of this section, we define the function $u_h(x) = -2/ac_0(1+(x/a)^2)$, which is the even solution decaying as $1/x^2$ at infinity of the Benjamin–Ono equation (see [4,7,2])

$$v + a\mathcal{H}(v') + ac_0v^2 = 0.$$

With the hypotheses made in the previous sections the following result holds:

Theorem 5.1. *Assume that* $\mathcal{L}_\varepsilon$ *satisfies* H1–H3 *and* H4 *and that* $\mathcal{N}_\varepsilon$ *satisfies* H5, H6 *and* H7. *Then for any* $0 < \alpha < 1$, *there exist* $\varepsilon_0 > 0$ *and* $u_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$ *and any* $0 \leqslant |u| < u_0\varepsilon$ (*if* $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$ *is independent of* k *then* $0 \leqslant |u| < u_0$), *Eq.* (5) *has a reversible solution* $U_{\varepsilon,u}$ *which satisfies*

$$U_{\varepsilon,u}(x) = \big(u + u_h(x)\big)\xi_0 + \widetilde{U}_{\varepsilon,u}(x),$$

where u_h *is the Benjamin–Ono homoclinic and where the term* $\widetilde{U}_{\varepsilon,u} \in B_2^{1,\alpha}(\mathbb{R})\xi_0 \oplus \mathcal{B}^\alpha(\widehat{\mathbb{D}})$ (*hence decaying as* $1/x^2$ *at infinity*) *is* $\mathrm{O}(\varepsilon + |u|\varepsilon^{-1})$. *Moreover, if* $D_U\mathcal{N}_\varepsilon(u\xi_0).\theta_k$ *is independent of* k, *then* $\widetilde{U}_{\varepsilon,u} = \mathrm{O}(\varepsilon + |u|)$.

In this theorem we state the existence of a family of solutions homoclinic to the stationary solutions $U(x) = u\xi_0$ for $|u|$ small enough. Notice that the decay rate when $x \to \pm\infty$ of these solutions is polynomial.

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