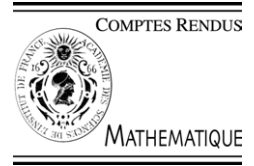




Available online at www.sciencedirect.com



C. R. Acad. Sci. Paris, Ser. I 339 (2004) 579–584



<http://france.elsevier.com/direct/CRASS1/>

Probability Theory

Stochastic Loewner evolution in multiply connected domains

Robert O. Bauer^{a,1}, Roland M. Friedrich^{b,2}

^a *Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA*

^b *Institute for Advanced Study, Princeton, NJ 08540, USA*

Received 12 August 2004; accepted 28 August 2004

Presented by Paul Malliavin

Abstract

We construct radial stochastic Loewner evolution in multiply connected domains, choosing the unit disk with concentric circular slits as a family of standard domains. The natural driving function or input is a diffusion on the associated moduli space. The diffusion stops when it reaches the boundary of the moduli space. We show that for this driving function the family of random growing compacts has a phase transition for $\kappa = 4$ and $\kappa = 8$, and that it satisfies locality for $\kappa = 6$. **To cite this article:** *R.O. Bauer, R.M. Friedrich, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

© 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Évolution stochastique de Loewner dans des domaines multiple connexes. Nous construisons l'évolution stochastique radiale de Loewner dans des domaines multiple connexes, en choisissant le disque d'unité avec des segments concentriques, comme famille de référence. La fonction naturelle qui fait croître les traces de l'évolution est une diffusion sur l'espace associé des modules. La diffusion s'arrête dès qu'il touche le bord de l'espace des modules. Nous démontrons que pour cette fonction qui engendre la croissance de ces compacts aléatoires, on trouve une transition de phase pour $\kappa = 4$ et $\kappa = 8$, et qu'ils satisfont la propriété de localité pour $\kappa = 6$. **Pour citer cet article :** *R.O. Bauer, R.M. Friedrich, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

© 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Version française abrégée

L'équation ordinaire de Loewner encode la croissance d'une courbe qui commence au bord du disque d'unité, et continue vers l'intérieur, par le mouvement associé selon le bord, i.e. du cercle. Pour les courbes aléatoires

E-mail addresses: rbauer@math.uiuc.edu (R.O. Bauer), rolandf@ias.edu (R.M. Friedrich).

¹ Research supported by NSA grant H98230-04-1-0039.

² Research supported by NSF grant DMS-0111298.

provenant par exemple d'un modèle de la mécanique statistique au point critique, Schramm [8] conclut de ces quelques propriétés de base, anticipées pour la limite d'échelle, en particulier, invariance conforme, une propriété de Markov, symétrie, continuité, que le mouvement au bord qui encode la courbe aléatoire doit être un changement linéaire de temps du mouvement Brownien standard. En inversant la procédure, c'est-à-dire en utilisant le mouvement brownien comme fonction qui fait pousser la courbe, il a obtenu une nouvelle classe de processus, nommé SLE_κ ($\kappa > 0$). C'est une idée naturelle d'essayer d'étendre les méthodes provenant du SLE aux cas des domaines autres que simplement connexes, comme l'on peut définir des modèles aléatoires sur ces domaines. Ainsi nous construisons l'évolution stochastique radiale de Loewner dans des domaines multiples connexes, en choisissant le disque d'unité avec des segments concentriques, comme famille de référence. En particulier, cela correspond à une généralisation et variation de l'équation de Komatu [1,2] au cas stochastique, et donne premièrement l'équation de Komatu–Loewner ordinaire

Théorème 0.1 (L'équation radiale de Komatu–Loewner). *La famille des application uniformisantes $\{f(z, t), 0 \leq t \leq T\}$ satisfait l'équation*

$$\frac{\partial \ln f(z, t)}{\partial t} = \frac{\partial F(\bar{\gamma}(t), w_t; t)}{\partial v} + \sum_{j=1}^{n-1} R_j(w_t; t) \frac{d \ln m_j(t)}{dt}, \quad (1)$$

avec la condition $f(z, T) = z$ pour tous $z \in D \setminus \{0\}$.

Comme le type conforme du domaine change en coupant le domaine selon le chemin, il n'est plus suffisant, pour inverser le processus, d'utiliser uniquement le mouvement sur le bord, par contre on doit ainsi inclure les modules, car les paramètres dépendent d'eux. En utilisant la méthode variationnelle de Schiffer nous obtenons un système associé d'équations pour les modules, qui nous permet d'inverser l'équation de Komatu–Loewner.

Théorème 0.2. *Pour un domaine standard D avec les paramètres (modules) M_1, \dots, M_{3n-4} et une fonction continue $t \mapsto \bar{\gamma}(t)$ avec des valeurs dans le cercle d'unité on peut résoudre (15) avec des valeurs initiales M_1, \dots, M_{3n-4} . La solution existe sur $(0, T]$. Si $T < \infty$, il existe un j tel que $\lim_{t \nearrow T} m_j(t) = 1$. Pour $z \in D$ et $t < T$, soit $g_t(z)$ la solution de (8) avec $g_0(z) = z$. La solution existe jusqu'en temps $T_z \in (0, T]$. Si $T_z < T$, alors $\lim_{t \nearrow T_z} (g_t(z) - \bar{\gamma}(t)) = 0$. Si K_t dénote la fermeture des points z tel que $T_z \leq t$, alors g_t applique $D \setminus K_t$ de manière biholomorphe sur le domaine standard D_t .*

Les assumptions concernant la nature de la courbe aléatoire (cf. ci-dessus) nous permet de conclure que la fonction qui engendre ces ensembles aléatoires, doit être une diffusion sur l'espace des modules des domaines avec des points marqués. La diffusion que nous définissons s'arrête dès qu'elle touche le bord de l'espace des modules. Nous démontrons que pour cette fonction qui engendre la croissance de ces compacts aléatoires, on trouve une transition de phase pour $\kappa = 4$ et $\kappa = 8$, et qu'ils satisfont la propriété de localité pour $\kappa = 6$.

1. Introduction

The stochastic Loewner evolution was introduced by Schramm in [8]. The Loewner equation shows how to encode a slit in the unit disk which starts at the boundary circumference in terms of a motion on the boundary. For a random slit such as an interface created by a statistical mechanics model at criticality, Schramm derived just from a few properties anticipated for the scaling limit of such a model—conformal invariance, a Markovian-type property, symmetry, continuity—that the random motion on the boundary encoding the random slit has to be a linear time-change of a standard Brownian motion. This new family of processes, SLE_κ ($\kappa > 0$), is studied in [3–6], and references therein. In particular, it was shown that a phase transition occurs at $\kappa = 4$ where the slit

ceases to be a simple curve and at $\kappa = 8$, where the slit becomes space-filling. For certain values of κ , SLE_κ has special properties—locality for $\kappa = 6$, and the restriction property for $\kappa = 8/3$.

Since the statistical mechanics models whose scaling limit is (or is conjectured to be) described by SLE_κ are readily defined for non-simply connected domains the question arises if the stochastic Loewner evolution can be extended to the non-simply connected case. The Loewner equation for simply connected domains has an analogue for multiply connected domains which we call the Komatu–Loewner equation. We derive a version of a result by Komatu for a different class of standard domains, [2]. Cutting a non-simply connected domain along a Jordan arc changes the conformal type of the domain. Because of this, the Komatu–Loewner equation is not enough to reverse the process, i.e. to begin with a driving function on the boundary and solve the equation—the coefficients in the equation depend on the conformal type of the domain. However, using Schiffer variations we obtain an associated system of equations for the conformal invariants. These equations, together with the Komatu–Loewner equation, allow us to reverse the process.

In the case of a random slit, the expected properties of the slit Schramm used for simply connected domains only allow us to conclude that the Komatu–Loewner equation should be driven by a diffusion on the space of conformal invariants, the moduli space. We define a class of diffusions on the moduli space for which the appropriate phase transitions occur at $\kappa = 4$ and $\kappa = 8$, and which satisfies locality at $\kappa = 6$.

2. Komatu–Loewner equation

Denote by D a domain in the complex plane bounded by a finite number $n \geq 2$ of proper continua. Then D is conformally equivalent to the unit disk cut along $n - 1$ disjoint concentric circular slits. We call such a domain a standard domain. If D is a standard domain we denote the boundary components by

$$C^{(j)} : |z| = m_j, \quad \theta_j \leq \arg z \leq \theta'_j, \quad 1 \leq j \leq n - 1, \quad C^{(n)} : |z| = 1. \tag{2}$$

Let \tilde{D} be a domain in the w -plane obtained from a standard domain by cutting along a slit (Jordan arc) Γ which starts from a point on the exterior boundary component $|w| = 1$ and avoids 0. Denote $w = f(z)$ the unique conformal map from a standard domain D onto \tilde{D} such that $f(0) = 0$, $f'(0) > 0$, and such that the peripheral boundary circumferences correspond to each other. Let $T = -\ln f'(0) > 0$. If we delete from the boundary of \tilde{D} a sub-arc of Γ with interior endpoint common with it, then there is a unique conformal map

$$w = h(w_t, t), \quad h(0, t) = 0, \quad h'(0, t) = e^{-t} \tag{3}$$

which maps a standard domain onto the domain thus obtained. Here $t \leq T$ and the peripheral boundary components are to correspond to each other. Define D_t by $h(D_t, t) = \tilde{D}$. Then D_t is a domain of the same type as \tilde{D} , i.e. obtained from a standard domain by cutting along a slit Γ_t . By the monotony property of the derivative $h'(0, t)$ the points on Γ are in one-to-one correspondence with the values of the parameter t , ranging over the interval $0 \leq t \leq T$. Let the conformal map from D onto D_t be denoted by

$$w_t = f(z, t), \quad f(0, t) = 0, \quad f'(0, t) = e^{t-T}. \tag{4}$$

Then $f(z) = f(z, 0) = h(f(z, t), t)$, and also $f(z, T) = z$. Further, let the boundary components of the circular slit disk $D_t + \Gamma_t$ be denoted by

$$C_t^{(j)} : |w_t| = m_j, \quad \theta_j(t) \leq \arg w_t \leq \theta'_j(t), \quad 1 \leq j \leq n - 1, \quad C_t^{(n)} : |w_t| = 1, \tag{5}$$

and the starting point on $C_t^{(n)}$ of the slit Γ_t be $\bar{\gamma}(t) = e^{-i\theta(t)}$. Denote $G(u, w_t; t)$ the Green function of the circular slit disk $D_t + \Gamma_t$ in the u -plane with pole at w_t , and denote a harmonic conjugate to $G(u, w_t; t)$ with respect to $w_t = x_t + iy_t$ by

$$H(u, w_t; t) = \int_0^{w_t} \left(\frac{\partial G(u, w_t; t)}{\partial x_t} dy_t - \frac{\partial G(u, w_t; t)}{\partial y_t} dx_t \right), \tag{6}$$

and set $F(u, w_t; t) = G(u, w_t; t) + iH(u, w_t; t)$. Denote $\omega_j(w_t; t)$ the harmonic measure of $C_t^{(j)}$ at w_t with respect to $D_t + \Gamma_t$ and let

$$R_j(w_t; t) = \frac{1}{2\pi} \int_{C_t^{(j)}} \frac{\partial F(u, w_t; t)}{\partial v} ds, \tag{7}$$

where $\partial/\partial v$ denotes differentiation at the boundary point u in the direction of the inner normal. We have $\Re(R_j) = \omega_j$ and we may assume that $\Im(R_j(0; t)) = 0$ for $j = 1, \dots, n - 1$.

Theorem 2.1 (Radial Komatu–Loewner equation). *The family $\{f(z, t), 0 \leq t \leq T\}$ satisfies the equation*

$$\frac{\partial \ln f(z, t)}{\partial t} = \frac{\partial F(\bar{\gamma}(t), w_t; t)}{\partial v} + \sum_{j=1}^{n-1} R_j(w_t; t) \frac{d \ln m_j(t)}{dt}, \tag{8}$$

with initial condition $f(z, T) = z$ for all $z \in D \setminus \{0\}$.

Note that, while the first and also the second term on the right in Eq. (8) are multiple-valued, their sum is single-valued. In the simply connected case the sum on the right-hand side of Eq. (8) disappears and the radial Komatu–Loewner equation reduces to the usual radial Loewner equation,

$$\frac{\partial \ln f(z, t)}{\partial t} = - \frac{f(z, t) + \bar{\gamma}(t)}{f(z, t) - \bar{\gamma}(t)}. \tag{9}$$

Remark 1. An alternative class of standard domains is provided by circular slit annuli. For that case Komatu [2] derived an equation analogous to (8). In the case $n = 2$ it can be written in terms of elliptic functions, [1].

If

$$\mathbf{P} = \mathbf{P}_t = [p_{j,k}(t)]_{j,k=1}^{n-1}, \quad p_{j,k}(t) = \int_{C_t^{(k)}} \frac{\partial \omega_j(w_t; t)}{\partial v} ds, \tag{10}$$

is the period matrix of D_t and $\lambda_j = -\ln m_j$, $\lambda^T = [\lambda_1, \dots, \lambda_{n-1}]$, then

$$\lambda^T = 2\pi \omega^T(0) \mathbf{P}^{-1}, \tag{11}$$

where $\omega^T = (\omega_1, \dots, \omega_{n-1})$ is the vector of harmonic measures. Under a Schiffer variation at the boundary point $\bar{\gamma}(t)$

$$\frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} 2\pi \omega^T(0) \mathbf{P}^{-1} = 2\pi \frac{\partial \omega(\bar{\gamma}(t), t)}{\partial v}^T \mathbf{P}^{-1}, \tag{12}$$

see [7]. Furthermore, the domain constant $d_n(t)$ defined by

$$\Re(\ln f^{-1}(w_t, t)) = \ln|w_t| + d_n(t) + O(|w_t|) \tag{13}$$

is given by $d_n(t) = T - t$. Under a Schiffer variation we have $\frac{\partial}{\partial \epsilon}|_{\epsilon=0} d_n(t) = -1$, see [7]. We find

$$\sum_{j=1}^{n-1} R_j(w_t; t) \frac{d \ln m_j(t)}{dt} = 2\pi \frac{\partial \omega(\bar{\gamma}(t), t)^T}{\partial v} \mathbf{P}_t^{-1} \mathbf{R}(w_t; t). \tag{14}$$

If $n > 1$, then the moduli space of an n -connected domain with one interior marked point is $3n - 4$ dimensional. Denote $M_l(t)$, $1 \leq l \leq 3n - 4$ the moduli of the domain $D_t + \Gamma_t$ with the origin as marked point.

Lemma 2.2. *There exist differentiable vector-fields $Y_l = Y_l(u, M_1, \dots, M_{3n-4})$, $1 \leq l \leq 3n - 4$, where u is complex of norm 1 and the M_k real such that if $M_1(t), \dots, M_{3n-4}(t)$ solve*

$$\dot{M}_l(t) = Y_l(\bar{\gamma}(t), M_1(t), \dots, M_{3n-4}(t)), \quad 1 \leq l \leq 3n - 4, \tag{15}$$

with initial values the parameters of the domain $D_0 + \Gamma_0$, then $\{f(z, t), 0 \leq t \leq T\}$ solves (8) with the Green function and harmonic measure at time t calculated from the parameters $M_k(t)$.

We can now reverse the procedure. Instead of beginning with a Jordan arc Γ that leads to a continuous curve $t \mapsto \bar{\gamma}(t)$ on the boundary circumference we begin with the curve $\bar{\gamma}(t)$.

Theorem 2.3. *For a standard domain D with parameters M_1, \dots, M_{3n-4} and a continuous function $t \mapsto \bar{\gamma}(t)$ with values in the unit circle we can solve (15) with initial values M_1, \dots, M_{3n-4} . The solution exists on $(0, T]$. If $T < \infty$, then there is a j such that $\lim_{t \nearrow T} m_j(t) = 1$. For $z \in D$ and $t < T$, let $g_t(z)$ be the solution of (8) with $g_0(z) = z$. The solution exists up to a time $T_z \in (0, T]$. If $T_z < T$, then $\lim_{t \nearrow T_z} (g_t(z) - \bar{\gamma}(t)) = 0$. If K_t denotes the closure of the set of points z for which $T_z \leq t$, then g_t maps $D \setminus K_t$ conformally onto a standard domain D_t .*

3. Diffusion on moduli space

To define SLE_κ on D we first define a diffusion on an appropriate moduli space. This is the moduli space \mathfrak{M}_n of an n -connected domain with one interior point and one boundary point marked. It has dimension $3n - 3$.

Let $\kappa > 0$ and denote B_t a standard one dimensional Brownian motion. Define $l(e^{i\theta}; t)$ by

$$l(e^{i\theta}; t) = \lim_{\varphi \rightarrow \theta} \left(\frac{\partial F(e^{i\theta}, e^{i\varphi}; t)}{\partial v} - \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}} \right). \tag{16}$$

Consider the system

$$d\theta(t) = \frac{1}{i} \left(l(\bar{\gamma}(t); t) dt + \sum_{j=1}^{n-1} R_j(\bar{\gamma}(t); t) d \ln m_j(t) \right) + \sqrt{\kappa} dB_t, \tag{17}$$

$$dM_l(t) = Y_l(\bar{\gamma}(t), M_1(t), \dots, M_{3n-4}(t)), \quad l = 1, \dots, 3n - 4, \tag{18}$$

where $\theta(0) = 0$, $\bar{\gamma}(t) = e^{i\theta(t)}$, and $M_l(0) = M_l$. We note again that the combination of terms in brackets in Eq. (17) is single-valued. The solution exists up to a stopping time τ . On the set $\tau < \infty$ we have for some j , $\lim_{t \nearrow \tau} m_j(t) = 1$. We call a solution to this system a *Schiffer diffusion (on moduli space)* and the random family of growing compacts K_t associated to a Schiffer diffusion via (8) *radial $n - SLE_\kappa$* . Alternatively, we may call the family of random conformal maps g_t *$n - SLE_\kappa$* . The following results show that the Schiffer diffusion has key properties we expect from our knowledge of SLE in simply connected domains.

Theorem 3.1. *If $\kappa \leq 4$, then K_t is a.s. a simple curve for $t < \tau$. If $\kappa > 4$ then K_t is a.s. not simple.*

This follows by considering the motion of points on the boundary.

Remark 2. We do not know if K_t can hit a circular slit in finite time with positive probability.

Theorem 3.2. *If $\kappa = 6$, then $n - SLE_\kappa$ satisfies locality.*

The proof is similar to the simply connected case. Our definition of the Schiffer diffusion was made with the previous two theorems in mind. We have not been able to confirm the restriction property for $\kappa = 8/3$. Instead we only obtained a family of martingales related to the restriction property. We will investigate in a forthcoming paper if the Schiffer diffusion is the essentially only class of diffusions for which these properties hold, and if restriction holds for $\kappa = 8/3$.

In a multiply connected domain there are two flavors of chordal SLE: Growing slits from a point on one boundary component to another point on the same boundary component, or growing slits from a point on one boundary component to a point on another boundary component. For the former we can repeat the above construction using as standard domain the upper half-plane with a finite number of horizontal slits, the compacts K_t growing toward ∞ . For the latter the construction can be based on Komatu's equation [2].

Acknowledgements

The authors would like to acknowledge the hospitality of the Institut für Stochastik at the University of Freiburg and of the Isaac Newton Institute where part of this work was done.

References

- [1] Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten, Proc. Phys. Math. Soc. Japan 25 (3) (1943) 1–42.
- [2] Y. Komatu, On conformal slit mapping of multiply-connected domains, Proc. Jpn. Acad. 26 (7) (1950) 26–31.
- [3] G.F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents. I. Half-plane exponents, Acta Math. 187 (2) (2001) 275–308.
- [4] G.F. Lawler, O. Schramm, W. Werner, Values of Brownian intersection exponents. II. Plane exponents, Acta Math. 187 (2) (2001) 237–273.
- [5] G.F. Lawler, O. Schramm, W. Werner, Conformal restriction: the chordal case, J. Amer. Math. Soc. 16 (4) (2003) 917–955 (electronic).
- [6] S. Rohde, O. Schramm, Basic properties of SLE, math.PR/0106036.
- [7] M. Schiffer, Hadamard's formula and variation of domain-functions, Amer. J. Math. 68 (1946) 417–448.
- [8] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. 118 (2000) 221–288.