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C. R. Acad. Sci. Paris, Ser. I 339 (2004) 637–642
COMPTES RENDUS MATHEMATIQUE
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Probability Theory

Geometry of foliations on the Wiener space and stochastic calculus of variations

Hélène Airault [a], Paul Malliavin [b], Jiagang Ren [c]

[a] *Laboratoire CNRS UMR 6140, LAMFA (Amiens), INSSET, université de Picardie Jules Verne, 48, rue Raspail, 02100 Saint-Quentin, France*
[b] *10, rue Saint-Louis en l'Isle, 75004 Paris, France*
[c] *Department of Mathematics, Zhongshan University, Guangzhou, Guangdong 510275, China*

Received 16 August 2004; accepted after revision 3 September 2004

Presented by Paul Malliavin

Abstract

Stochastic Calculus of variations deals with maps defined on the Wiener space, with finite dimensional range; within this context appears the notion of *non-degenerate map,* which corresponds roughly speaking to some kind of infinite dimensional ellipticity; a non-degenerate map has a smooth law; by conditioning, it generates a continuous desintegration of the Wiener measure. Infinite dimensional Stochastic Analysis and particularly SPDE theory raise the natural question of what can be done for maps with an infinite dimensional range; our approach to this problem emphasizes an intrinsic geometric aspect, replacing range by generated σ-field and its associated foliation of the Wiener space. ***To cite this article: H. Airault et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Géométrie des foliations sur l'espace de Wiener et calcul des variations stochastiques. Le Calcul Stochastique des variations considère classiquement des applications de l'espace de Wiener dans un espace de dimension finie ; dans ce contexte s'inscrit la théorie des *applications non dégénérées* pour lesquelles on peut établir la régularité des lois ainsi que l'existence de désintégrations continues. L'Analyse stochastique en dimension infinie et singulièrement la théorie des SPDE, pose la question naturelle de l'étude des applications de l'espace de Wiener dans un espace de dimension infinie. Nous approchons ce problème de manière intrinsèque, privilégiant l'étude géomètrique des sous tribus à travers leurs foliations associées. ***Pour citer cet article : H. Airault et al., C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

E-mail addresses: helene.airault@insset.u-picardie.fr (H. Airault), sli@ccr.jussieu.fr (P. Malliavin), mcdm06@zsu.edu.cn (J. Ren).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.09.009

1. Introduction: the theory of non-degenerate maps

We work on an abstract Wiener space X, μ is its canonical Wiener measure; denote by H the Cameron–Martin space, that is the Hilbert space of constant vector fields for which the measure μ is quasi-invariant under translation by a vector of H. Denote by $D_r^p(X)$ the Sobolev space of functions which are r-times differentiable relatively to H, with all their derivatives belonging to L_μ^p. Let $D_r^{\infty^-}(X) := \bigcap_{p<\infty} D_r^p(X)$. Consider a map $g:\ X \mapsto R^d,\ g = (g_1,\dots,g_d)$. Assume that $g_i \in D_2^{\infty^-}(X)$; then define the $d\times d$ covariance matrix $\gamma_{i,j} = (\nabla g_i|\nabla g_j)$; the map g is said to be non-degenerate, if almost surely the matrix $\gamma_{*,*}$ is invertible and if, denoting $\gamma^{*,*}$ its inverse matrix, then $\gamma^{i,j} \in L_\mu^{\infty^-}$, an hypothesis which implies that $\gamma^{i,j} \in D_1^{\infty^-}(X)$. The canonical lift to X of the coordinate vector field $\frac{\partial}{\partial\xi_s}$ is defined as

$$Z_{\mathrm{can}}^s(\omega) = \sum_{i=1}^d \gamma^{si}(\omega)(\nabla g_i)(\omega), \quad \text{we have} \quad (Z_{\mathrm{can}}^j|Z_{\mathrm{can}}^s) = \gamma^{js}. \tag{1}$$

The hypothesis: g non-degenerate implies that $Z^s \in D_1^{\infty^-}(X;H)$ and Z^s *has a divergence relatively to the Gaussian measure* μ, let $\delta_\mu(Z^s) \in L_\mu^{\infty^-}(X)$.

$$\int \delta_\mu(Z^s)\psi\,\mathrm{d}\mu = -\int (Z^s|\nabla\psi)\,\mathrm{d}\mu \quad \text{for any differentiable function } \psi \text{ on } X. \tag{2}$$

Denote $\nu = g_*(\mu)$ the law of g, $g_*(\mu)(A) = \mu(g^{-1}(A))$. Then the divergence of $\frac{\partial}{\partial\xi_s}$ relative to ν is the conditional expectation of the divergence of Z^s relatively to μ,

$$\delta_\nu\left(\frac{\partial}{\partial\xi_s}\right) = E^g\big(\delta_\mu(Z^s)\big) \tag{3}$$

consequently $\delta_\nu(\frac{\partial}{\partial\xi_s}) \in L_\nu^{\infty^-}$, and ν has an Hölderian density relatively to the Lebesgue measure. More generally, we call a lift by g of $\frac{\partial}{\partial\xi_s}$, a vector field Z^s such that

$$Z^s(h\,o\,g) = \left(\frac{\partial}{\partial\xi_s}h\right)o\,g. \tag{4}$$

It satisfies (4) if and only if $(Z^s|\nabla g_s) = 1$ and $(Z^s|\nabla g_k) = 0$ if $k \neq s$. Moreover, there is a unique lift by g of $\frac{\partial}{\partial\xi_s}$, in the space generated by $(\nabla g_j)_{j=1,\dots,d}$, this is given by Z_{can}^s.

Conditioning by a non-degenerate map preserves differentiability ([5]; page 82). For any lift Z^s by g of $\frac{\partial}{\partial\xi_s}$, and such that the divergence $\delta_\mu(Z_s)$ exists, we have

$$\frac{\partial}{\partial\xi_s}E^g(f) - E^g(D_{Z^s}f) = E^g\big(\delta_\mu(Z^s)f\big) - E^g\big(\delta_\mu(Z^s)\big)\times E^g(f), \quad \forall f \in D_1^{\infty^-}(X). \tag{5}$$

With (3) and (5), divergences of lifts play a key role. An analytic expression of the divergence of the canonical lift has been given in coordinates in [5], page 71. In this Note, we give alternative geometric expressions free from coordinates which could be used in the case of σ-fields (see [3,4,6]).

2. Lifting and Lie derivatives

A vector field Z on X is called *basic* if $Z \in D^{\infty^-}(X;H)$ and if $D_Z g_s = (Z|\nabla g_s)$ is g-measurable for all $s \in [1,d]$. Thus, if Z is basic,

$$\big(Z(\omega)|\nabla g_s(\omega)\big) = z_s\big(g(\omega)\big), \quad s = 1,\dots,d. \tag{6}$$

A vector field Z on X is called g-basically constant if $D_Z g_s$ is constant for any $s \in [1, d]$. In particular, any lift by g of a constant vector field is g-basically constant. Let $\phi \in R^d$, then a lift by $\phi\, o\, g$ of a constant vector field is basic. If a basic vector field Z is in the normal space, then (compare with (1)),

$$Z(\omega) = \sum_k z_k\big(g(\omega)\big) Z^k_{\text{can}}(\omega). \tag{7}$$

In the following, unless it is precised, for lift, we mean a lift by g. Let $\varpi = \mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_d$.

Proposition 2.1. *The Lie derivative of ϖ relatively to any basically constant vector field vanishes.*

Proof. We use the Cartan formula $\mathcal{L}_Z = i(Z)\,\mathrm{d} + \mathrm{d}i(Z)$, where $i(Z)$ denotes the interior product by Z. We have $\mathcal{L}_Z(\mathrm{d}g) = \mathrm{d}(D_Z g)$. As $\mathrm{d}\varpi = 0$, the first term of Cartan formula disappears. We obtain $i(Z)\varpi = (D_Z g_1) \times \mathrm{d}g_2 \wedge \cdots \mathrm{d}g_d + \cdots$, expression which has a vanishing boundary. □

2.1. The duality

We denote by Ω the differential form of degree ∞ which is the 'volume form' associated to the Gaussian measure μ. Let $\mathcal{C}$ be the differential form of degree $\infty - d$ given by $\mathcal{C} = i(\varpi)\Omega$, i.e. $\mathcal{C} \wedge \varpi = \Omega$.

Proposition 2.2. *The unnormalized conditional expectation is given by $E^g(f)(\xi) = \int_{g^{-1}(\xi)} f \times \mathcal{C}$.*

Proof. For $f : X \to R$, the unnormalized conditional expectation is given by (see [1])

$$E^g[f](\alpha) = \int\limits_{g^{-1}(\alpha)} f(\omega)\sqrt{\det\big(\gamma^{**}(\omega)\big)}\,\mathrm{d}a(\omega), \quad \alpha \in R^d.$$

Denote $|\varpi| = \sqrt{\det((\nabla g_s|\nabla g_k))}$. The $\infty - d$ Gaussian area $\mathrm{d}a$ satisfies $\mathrm{d}a \wedge \frac{\varpi}{|\varpi|} = \Omega$. Thus $\frac{\mathrm{d}a}{|\varpi|} \wedge \varpi = \Omega$. As $\frac{\mathrm{d}a}{|\varpi|}$ corresponds to the conditional law, the proposition is proved. □

Theorem 2.3. *Let Z be a basically constant vector field, then the Lie derivative $\mathcal{L}_Z\mathcal{C} = \delta_\mu(Z) \times \mathcal{C}$.*

Proof. $\delta_\mu(Z) \times \Omega = \mathcal{L}_Z(\Omega) = \mathcal{L}_Z(\mathcal{C}) \wedge \varpi + \mathcal{C} \wedge \mathcal{L}_Z\varpi = \mathcal{L}_Z(\mathcal{C}) \wedge \varpi$. □

Remark 1. Theorem (2.3) reduces the derivative of a conditional expectation to a Skorokhod integral on the Wiener space. With Proposition 2.2, it gives a geometrical proof of (5).

3. Canonical forms of a foliation: curvature and proximity

To avoid the difficulty of selecting versions of maps, we consider the case where X is a finite dimensional Euclidean space with its canonical Gaussian measure; then by Sobolev embedding, $D_1^{\infty^-}(X)$ is contained in the set of continuous functions; we always take the continuous versions. Since the estimates that we obtain are independent of the dimension of X, they stay valid for the infinite dimensional case, that is the Wiener space. Let $g = (g_1, g_2, \ldots, g_d) : X \to R^d$ be a non-degenerate map. On X, we define the equivalence relation $\omega_1 \mathcal{T} \omega_2$ if $g(\omega_1) = g(\omega_2)$. Since $g(\omega_1) = g(\omega_2)$ implies that $\phi(g(\omega_1)) = \phi(g(\omega_2))$ for any diffeomorphism ϕ of R^d, the relation $\mathcal{T}$ depends only on the σ-field generated by the $\phi\, o\, g$, $\phi \in \mathrm{diff}(R^d)$. Let N_ω be the normal suspace at ω: it is the d-dimensional subspace of X generated by $(\nabla g_j(\omega))_{j=1,\ldots,d}$. Since (1), the canonical lifts $(Z^j_{\text{can}})_{j=1,\ldots,d}$ form a basis of N_ω.

Proposition 3.1. *Consider the normal spaces N_{ω_1} and N_{ω_2} at two different points ω_1 and ω_2. There exists a unique linear mapping $\tau_{\omega_2\leftarrow\omega_1}: N_{\omega_1}\to N_{\omega_2}$ such that $\tau_{\omega_2\leftarrow\omega_1}(n_1)=n_2$ is given by*

$$\langle dg_k(\omega_1), n_1\rangle = \langle dg_k(\omega_2), n_2\rangle \quad \textit{for } k=1,\dots,d. \tag{8}$$

With the basis $(\nabla g_j(\omega_1))_{j=1,\dots,d}$ for N_{ω_1} and $(\nabla g_j(\omega_2))_{j=1,\dots,d}$ for N_{ω_2}, the matrix of $\tau_{\omega_2\leftarrow\omega_1}$ is $\gamma^{**}(\omega_2)\gamma_{**}(\omega_1)$. Because of (4), the image by $\tau_{\omega_2\leftarrow\omega_1}$ of the canonical lift $Z^j_{\mathrm{can}}(\omega_1)$ is $Z^j_{\mathrm{can}}(\omega_2)$. If $g(\omega_1)=g(\omega_2)$, then $\tau_{\omega_2\leftarrow\omega_1}$ is the same for any $\phi\, o\, g$, when $\phi\in R^d$. For $h\in H$, let

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} Z^s_{\mathrm{can}}(\omega+\epsilon h) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0}\gamma^{sj}(\omega+\epsilon h)\gamma_{jp}(\omega)\, Z^p_{\mathrm{can}}(\omega) + \gamma^{sj}\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0}\nabla g_j(\omega+\epsilon h). \tag{9}$$

Since $(Z^s_{\mathrm{can}}|\nabla g_s)=1$, we obtain

$$2\sum_s\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} Z^s_{\mathrm{can}}(\omega+\epsilon h)\Big|\nabla g_s(\omega)\right) = \sum_{s,j}\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0}\gamma^{sj}(\omega+\epsilon h)\gamma_{js}(\omega). \tag{10}$$

For $\omega\in X$, $X=R^m$ and $h\in H$, $H=R^m$, consider the matrix

$$\beta(\omega;h) = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\gamma^{**}(\omega+\epsilon h)\right]_{|\epsilon=0}\gamma_{**}(\omega) = -\gamma^{**}(\omega)\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\gamma_{**}(\omega+\epsilon h)\right]_{|\epsilon=0}. \tag{11}$$

The map $(\omega,h)\to\beta(\omega;h)$ from $X\times H\to \mathrm{End}(R^d)$ is called the *proximity form* of the foliation g. We define $\nabla^2 g_j(\omega): H\times H\to R$ by $\nabla^2 g_j(\omega)(Y,Z)=\sum_{k,p}\frac{\partial^2 g_j}{\partial\xi_k\,\partial\xi_p}(\omega)Y_k Z_p$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\gamma_{**}(\omega+\epsilon h)_{|\epsilon=0} = \left\{\nabla^2 g_i(\omega)\big(h,\nabla g_j(\omega)\big) + \nabla^2 g_j(\omega)\big(h,\nabla g_i(\omega)\big)\right\}_{ij}. \tag{12}$$

The orthogonal projection on N_ω of (9) is $\mathrm{Proj}_{N_\omega}\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} Z^s_{\mathrm{can}}(\omega+\epsilon h)=\sum_r \alpha_{sr}(\omega)Z^r_{\mathrm{can}}(\omega)$ with

$$\alpha_{sr}(\omega) = \beta(\omega;h)_{sr} + \sum_j \nabla^2 g_j(h,\nabla g_r)\gamma^{js} = -\nabla^2 g_r(h, Z^s_{\mathrm{can}}). \tag{13}$$

The tangent space T_ω is the subspace of $h\in H$ such that $(\nabla g_k(\omega)|h)=0$ for $k=1,\dots,d$.

Theorem 3.2. *For $h\in H$,*

$$\sum_s\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0} Z^s_{\mathrm{can}}(\omega+\epsilon h)\Big|\nabla g_s(\omega)\right) = -\sum_j\nabla^2 g_j(\omega)\big(Z^j_{\mathrm{can}}(\omega),h\big) = \frac12\,\mathrm{trace}\,\beta(\omega;h). \tag{14}$$

and for $h\in T_\omega$, (14) *is the same for any $\phi\, o\, g$, when ϕ varies in* $\mathrm{diff}(R^d)$.

From (13), we see that taking the derivative of the map $\tau_{\omega_2\leftarrow\omega_1}$ tangentially to a leaf, defines a connection on the normal fiber bundle of the foliation. This connection has a curvature equal to zero and it is integrable. The proximity form is the sum of this connection with the transposed connection.

Remark 2. In (14), if $h\notin T_\omega$, then for $\phi=(\phi_1,\phi_2,\dots,\phi_d)\in\mathrm{diff}(R^d)$,

$$\sum_j\nabla^2(\phi_j\, o\, g)(\omega)(Z^{\phi_j\, o\, g}_{\mathrm{can}},h) = \sum_j\nabla^2 g_j(Z^{g_j}_{\mathrm{can}},h) + \big(\nabla\log(\det{}_{jac}\phi)\big(g(\omega)\big)|h\big).$$

For non-degenerate $g: X\to R^d$, the second fundamental form at ω is (see [2])

$$l^g_\omega(h_1,h_2) = \sum_{r,s}\gamma^{rs}\nabla^2 g_r(h_1,h_2)\nabla g_s = \sum_r\nabla^2 g_r(h_1,h_2)Z^r_{\mathrm{can}}. \tag{15}$$

If h_1 or h_2 is in the tangent space T_ω, then $l_\omega^{\phi\, o\, g}(h_1,h_2)=l_\omega^g(h_1,h_2)$ for any diffeomorphism ϕ of R^d. In that case, we denote $l_\omega(h_1,h_2)$. Consider

$$L_\omega^g = \mathrm{trace}_{H\times H}\, l_\omega^g(\bullet,\bullet)=\sum_r\left(\sum_k \frac{\partial^2 g_r}{\partial \xi_k^2}\right)Z_{\mathrm{can}}^r. \tag{16}$$

From (1) and (16), it holds $(L_\omega^g|Z_{\mathrm{can}}^s(\omega))=\sum_{i,k}\gamma^{si}(\omega)\frac{\partial^2 g_i}{\partial\xi_k^2}(\omega)$ and $(L_\omega^g|\nabla g^s(\omega))=\sum_k \frac{\partial^2 g_s}{\partial \xi_k^2}(\omega)$.

Theorem 3.3. $\sum_{jk}\nabla^2 g_k(\omega)(Z_{\mathrm{can}}^j,\nabla g_j(\omega))Z_{\mathrm{can}}^k=\mathrm{trace}_{N_\omega\times N_\omega}\, l_\omega^g(\bullet,\bullet)$.

We define the total curvature of the leaf at ω as

$$\mathcal{C}_\omega=\mathrm{trace}_{T_\omega\times T_\omega}\, l_\omega(\bullet,\bullet)=L_\omega^g-\mathrm{trace}_{N_\omega\times N_\omega}\, l_\omega^g(\bullet,\bullet) \tag{17}$$

The proximity form is given in terms of the trilinear form

$$\alpha_\omega^g(h_1,h_2,h_3)=-\sum_{p,s}\gamma^{ps}(\omega)\big(l_\omega^g\big(h_1,\nabla g_p(\omega)\big)|h_2\big)\big(\nabla g_s(\omega)|h_3\big). \tag{18}$$

If $h_1\in T_\omega$, then for any diffeomorphism ϕ of R^d, $\alpha_\omega^{\phi\, o\, g}(h_1,h_2,h_3)=\alpha_\omega^g(h_1,h_2,h_3)$.

4. The canonical form of a foliation

For $m>d$, we consider differentiable maps $g=(g_1,g_2,\dots,g_d):R^m\to R^d$, non-degenerate at 0, i.e. the matrix $\gamma_{**}(0)$ is invertible. We define the equivalence relations:

(1) $g_1\sim g_2$ if there exists ϕ, a diffeomorphism of R^d, $\phi\in\mathrm{diff}(R^d)$ such that $g_2=\phi\, o\, g_1$.

(2) $g_1\vee g_2$ if $\|g_1(\xi)-g_2(\xi)\|\leqslant \textit{constant}\times\|\xi\|^3$ when $\|\xi\|\to 0$.

Let $(\sim,\vee)$ be the equivalence relation on the set of differentiable maps from R^m to R^d, obtained by superposing the two equivalence relations $\sim$ and $\vee$.

Theorem 4.1. *In any equivalence class for $(\sim,\vee)$, there is a unique map $\xi\to\eta(\xi)$ from R^m to R^d with the following properties: There exist an orthonormal basis $(e_j)_{j=1,\dots,m}$ of R^m, for $\xi\in R^m$, $\xi=\sum_{j=1}^d\hat{\hat{\xi}}_j e_j+\sum_{j=d+1,\dots,m}\hat{\xi}_j e_j$, and an orthonormal basis $(f_j)_{j=1,\dots,d}$ of R^d, for $\eta\in R^d$, $\eta=\sum_{j=1,\dots,d}\eta_j f_j$ such that for $i=1,\dots,d$, we have $\eta_i(\xi)=\hat{\hat{\xi_i}}+q_i(\hat{\xi})+B_i(\hat{\hat{\xi}},\hat{\xi})$ where $\hat{\xi}=(\overbrace{0,\dots,0}^{d\ times},\widehat{\xi_{d+1}},\dots,\widehat{\xi_m})$ and $\hat{\hat{\xi}}=(\hat{\hat{\xi}}_1,\dots,\hat{\hat{\xi}}_d,\overbrace{0,\dots,0}^{m-d\ times})$. q_i is an homogeneous polynomial of degree 2 and B_i is a bilinear form.*

This reduction is called the *canonical form of the foliation*. The equivalence relation $(\sim,\vee)$ respects the Euclidean structure of R^m and that of R^d. For the study of laws of random variables defined on the Wiener space X, the Euclidean structure of H is relevant, while on the image space R^d, only the volume structure is needed. For $\omega\in R^m$, let

$$g_k(\omega)=\hat{\hat{\xi}}_k+q_k(\hat{\xi})+B_k(\hat{\hat{\xi}},\hat{\xi}),\quad k=1,\dots,d,\ \omega=(\hat{\hat{\xi}},\hat{\xi}),\ \hat{\hat{\xi}}\in R^d,\ \hat{\xi}\in R^{m-d}. \tag{19}$$

The vector field ∇g_k has for components

$$[\nabla g_k]^s(\omega)=\delta_k^s+B_k(e_s,\hat{\xi}),\quad s\in[1,d];\qquad [\nabla g_k]^s(\omega)=\frac{\partial}{\partial\hat{\xi}_s}q_k(\hat{\xi})+B_k(\hat{\hat{\xi}},e_s),\quad s\in\,]d,n], \tag{20}$$

where $(e_1,\dots,e_d,e_{d+1},\dots,e_m)$ is the canonical basis of R^m. Thus $\gamma_{**}(0)=I_{R^d}$. The divergence $\delta\nabla g_k$ of ∇g_k relatively to the Lebesgue volume is

$$-\operatorname{trace}_{H\times H}\nabla^2 g_k(0)(\bullet,\bullet) = -\sum_s \frac{\partial}{\partial\xi_s}[\nabla g_k]^s = -\sum_{s>d}\frac{\partial^2 q_k}{\partial\hat{\xi}_s^2}. \tag{21}$$

The normal space at 0 is the set of $\hat{\hat{\xi}} \in R^d$ and the tangent space at 0 is the set of $\hat{\xi} \in R^{m-d}$. The second fundamental form of g at $\xi = 0$ is $(q_i(\hat{\xi}))_{i=1,\dots,d}$. The curvature $\mathcal{C}_0$ of the leaf $g(\omega) = g(0)$ is given by (21). We define $\rho(\hat{\xi}) \in \operatorname{End}(R^d)$ by

$$\rho(\hat{\xi})(\hat{\hat{\xi}}) = \big(B_j(\hat{\hat{\xi}},\hat{\xi})\big)_{j=1,\dots,d} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}_{|\epsilon=0}\tau_{0+\epsilon h\leftarrow 0} \quad \text{where } h = (\hat{\hat{\xi}},\hat{\xi}). \tag{22}$$

Then $(\beta(0;\hat{\xi}))_{jk} = -[\rho(\hat{\xi})_{jk} + \rho(\hat{\xi})_{kj}]$ and $(\rho(\hat{\xi}))_{jk} = B_j(e_k,\hat{\xi}) = \nabla^2 g_j(\hat{\xi},\nabla g_k(0))$.

Remark 3. For $j = 1,\dots,d$, we can obtain a lift Z^j_{prox} of $\frac{\partial}{\partial\eta_j}$ using the proximity endomorphism: Let $g : X \to R^d$ defined by $\eta_i(\xi) = \hat{\hat{\xi}}_i + q_i(\hat{\xi}) + [\rho_i(\hat{\xi})(\hat{\hat{\xi}})]_i$, for $i = 1,\dots,d$ and consider the matrix $m_{kj} = ((I+\rho(\hat{\xi}))^{-1})_{kj}$, then

$$Z^j_{\mathrm{prox}} = \sum_{k=1}^d m_{kj}(\hat{\xi})\frac{\partial}{\partial\hat{\hat{\xi}}_k} \text{ are respectively lifted vector fields of } \partial_{\eta_j}, \quad j = 1,\dots,d. \tag{23}$$

The vector fields $(Z^j_{\mathrm{prox}}(\xi))$ have a simple expression, but they are not in the normal space N_ξ and they are not defined everywhere since the matrix $(I+\rho(\hat{\xi}))$ is not always invertible. For the canonical lifts, this difficulty does not occur.

With (11)–(17), and (21), (22), we obtain the divergences of canonical lifts as trace of the curvature and proximity forms.

Theorem 4.2. *For the gradient vector fields, it holds* $\delta_\mu(\nabla g^s)(\omega) = (\omega - \mathcal{C}_\omega|\nabla g^s(\omega))$ *and for the canonical lifts* (1),

$$\delta_\mu(Z^s_{\mathrm{can}})(\omega) = (\omega - \mathcal{C}_\omega|Z^s_{\mathrm{can}}) - \frac{1}{2}\operatorname{trace}\beta(\omega; Z^s_{\mathrm{can}}), \tag{24}$$

where $\operatorname{trace}\beta(\omega; Z^s_{\mathrm{can}})$ *is given by* (14).

For a basic vector field Z in the normal space, see (7), the formula (24) extends with the additional term $\delta_\nu(z)$, the divergence with respect to $\nu = g * \mu$ of the vector field $z = \sum_j z_j\partial_j$ on R^d deduced from (7). In infinite dimension, the scalar product $(x|Z)$ involves a Skorokhod stochastic integral.

Acknowledgement

This work was completed while J. Ren visited INSSET, July 2004.

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