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C. R. Acad. Sci. Paris, Ser. I 339 (2004) 893-898



http://france.elsevier.com/direct/CRASS1/

Numerical Analysis

A finite volume method for solving Maxwell equations in inhomogeneous media on arbitrary meshes

François Hermeline

CEA/DIF, DSSI/SNEC, BP 12, 91680 Bruyères le Châtel, France Received 25 June 2004; accepted 27 September 2004 Available online 13 November 2004 Presented by Olivier Pironneau

Abstract

We present a new finite volume method for solving Maxwell equations in inhomogeneous media. This method has several advantages: (i) it allows even distorted or non-convex arbitrary polygonal meshes to be used; (ii) it preserves the Gauss law; (iii) it leads to an explicit differential system; (iv) it generalizes the standard finite difference method and the finite volume method on Delaunay–Voronoi meshes. *To cite this article: F. Hermeline, C. R. Acad. Sci. Paris, Ser. I 339 (2004).* © 2004 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Une méthode de volumes finis pour les équations de Maxwell en milieu inhomogène sur des maillages arbitraires. On présente une nouvelle méthode d'approximation du type volumes finis pour les équations de Maxwell en milieu inhomogène. Cette méthode possède plusieurs avantages : (i) elle permet d'utiliser des maillages de polygones quelconques même très déformés ou non convexes ; (ii) elle préserve la loi de Gauss ; (iii) elle fournit un système differentiel explicite ; (iv) elle généralise la méthode des différences finies usuelle et les méthodes de volumes finis sur des maillages de Delaunay–Voronoi. *Pour citer cet article : F. Hermeline, C. R. Acad. Sci. Paris, Ser. I 339 (2004).*

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Version française abrégée

Il est bien connu que le schéma aux differences finis décrit dans [9] est une des méthodes les plus employées pour discrétiser les équations de Maxwell. Par ailleurs on a déjà vu l'intérêt d'utiliser les propriétés d'orthogonalité entre les maillages de Delaunay et de Voronoi pour généraliser ce type de schéma à d'autres maillages que des maillages de rectangles : voir [4] par exemple.

E-mail address: francois.hermeline@cea.fr (F. Hermeline).

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On se propose dans ce travail de décrire une méthode de volumes finis qui permet de s'affranchir de la condition pour le maillage primal d'être de Delaunay, et même d'utiliser n'importe quel maillage primal, éventuellement très déformé, non convexe ou non conforme, pour discrétiser les équations de Maxwell en dimension deux.

Considérons, par exemple, le mode de polarisation électrique transverse. Le principe de la méthode, qui dérive de [5–7], est d'intégrer deux fois les équations scalaires (lois de Faraday et de Gauss) : une fois sur le maillage primal et une fois sur un maillage dual que l'on définit içi en joignant les barycentres des mailles primales et les milieux de leurs côtés. On obtient ainsi les valeurs du champ magnétique aux barycentres et aux sommets des mailles primales. A partir de ces valeurs on peut calculer une estimation correcte, à l'aide de l'équation d'Ampère–Maxwell, des composantes tangentielles du champ électrique le long des côtés des mailles primales et duales. Il faut noter que la loi de Gauss discrétisée reste préservée au cours du temps pourvu qu'elle soit vérifiée au temps initial et pourvu, bien entendu, que la relation de conservation de la charge discrétisée soit vérifiée.

On donne ensuite un exemple simple d'application au cas du calcul des modes propres d'une cavité carrée constituée de deux matériaux distincts et on compare les résultats obtenus avec un maillage de carrés et un maillage de quadrilatères très déformés.

La méthode peut évidemment être utilisée pour des équations du même type comme celles de l'acoustique, par exemple. Elle peut aussi être aisément généralisée au cas des milieus anisotropes.

1. Definitions and notation

Given Ω a bounded two-dimensional polygonal domain, we use a mesh on Ω (called primal mesh) made up of arbitrary polygons. With each (primal) element *P* of this mesh we associate one (primal) point *p*: the centroid is a qualified candidate but other points can be chosen. By connecting the primal points and the midpoints *f* of the sides we obtain a second mesh on Ω (called dual mesh). With each (dual) element Π of this mesh we can associate one vertex of the primal mesh that will be denoted by *d*. Furthermore let us denote by (Fig. 1):

- $P_p(\Pi_d)$ the primal (dual) polygon associated with p(d);
- $\|P_p\| (\|\Pi_d\|)$ the area of $P_p (\Pi_d)$;
- $F_f = [d, e] (\Phi_{fp} = [p, f], \Phi_{fd} = [f, q])$ the side of a primal (dual) polygon;
- $-|F_f|(|\Phi_{fp}|, |\Phi_{fd}|)$ the length of $F_f(\Phi_{fp}, \Phi_{fq})$;
- *t* the unit counterclockwise tangent vector on the boundary $\partial \Omega$ of Ω ;
- \mathbf{n}_{f} the unit outward normal vector on the side F_{f} of P_{p} ;
- $-t_f$ the unit counterclockwise tangent vector on the side F_f of P_p ;
- $\mathbf{v}_{fp} (\mathbf{v}_{fq})$ the unit outward normal vector on the side $\Phi_{fp} (\Phi_{fq})$ of Π_d ;
- $\tau_{fp} (\tau_{fq})$ the unit clockwise tangent vector on the side $\Phi_{fp} (\Phi_{fq})$ of Π_d ;
- $-\theta_{fp}$ (θ_{fq}) the angle between t_f and v_{fp} (v_{fq}).

2. Approximation of Maxwell equations

For example consider Maxwell equations in the transverse electric mode of polarization with a perfect conductor boundary condition (ε , μ being arbitrary discontinuous positive coefficients):

$$\begin{cases} \frac{\partial B}{\partial t} + \nabla \times E = 0, & \frac{\partial D}{\partial t} - \nabla \times H = -j, \quad \nabla \cdot D = \rho \quad \text{in } \Omega \times [0, T], \\ D = \varepsilon E, \quad B = \mu H \quad \text{in } \Omega \times [0, T], \\ E \cdot t = \mathbf{0} \quad \text{in } \partial \Omega \times [0, T]. \end{cases}$$
(1)

F. Hermeline / C. R. Acad. Sci. Paris, Ser. I 339 (2004) 893-898



Fig. 1. Left: a sample primal mesh (solid lines) and its dual mesh (dashed lines). Right: two primal (dual) polygons sharing the side $F_f = [d, e]$ $(\Phi_{fp} \cup \Phi_{fq} = [p, f] \cup [f, q]).$

Recall that the third equation (Gauss law) is satisfied at any time, provided it is satisfied at the initial time and if the following charge conservation law holds:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \quad \text{in } \boldsymbol{\Omega} \times [0, T].$$

By drawing inspiration from [5–7] we make the following operations:

- integrate the first equation over the primal polygon P_p ,
- integrate the first equation over the dual polygon $\Pi_d \cap P_p$ and add it for all polygons P_p such that $\Pi_d \cap P_p \neq \emptyset$,
- take the flux of the second equation over the primal and dual sides F_f and Φ_{fp} , Φ_{fq} ,
- integrate the third equation over the primal and dual polygons P_p and Π_d .

Let be:

- $-\varepsilon_p, \mu_p, \rho_p$ approximations of ε, μ, ρ in P_p ,
- H_p an approximation of H in P_p which provides an approximation $B_p = \mu_p H_p$ ($B_{dp} = \mu_p H_d$) of B in P_p ($\Pi_d \cap P_p$),
- H_d , ρ_d approximations of H, ρ in Π_d ,
- H_f , j_f approximations of H, j at the point f,
- $E_{fp}(E_{fq})$ an approximation of E in $P_p(P_q)$ in the neighborhood of F_f which provides an approximation $D_{fp} = \varepsilon_p E_{fp} (D_{fq} = \varepsilon_q E_{fq})$ of D.

Since the tangential (normal) component of E(D) is continuous, the following conditions must be satisfied:

$$\boldsymbol{E}_{fp} \cdot \boldsymbol{t}_{f} = \boldsymbol{E}_{fq} \cdot \boldsymbol{t}_{f} \stackrel{\text{def}}{=} \boldsymbol{E}_{f} \cdot \boldsymbol{t}_{f} \quad \text{and} \quad \varepsilon_{p} \boldsymbol{E}_{fp} \cdot \boldsymbol{n}_{f} = \varepsilon_{q} \boldsymbol{E}_{fq} \cdot \boldsymbol{n}_{f} \stackrel{\text{def}}{=} \boldsymbol{D}_{f} \cdot \boldsymbol{n}_{f}.$$
(2)

Then from (1) we obtain:

$$\begin{cases} \|P_p\| \frac{\partial B_p}{\partial t} + \sum_{f \in \partial P_p} |F_f| E_f \cdot t_f = 0, \\ \sum_p \|\Pi_d \cap P_p\| \frac{\partial B_{dp}}{\partial t} - \sum_{f \in \partial \Pi_d} \left(|\Phi_{fp}| E_{fp} \cdot \tau_{fp} + |\Phi_{fd}| E_{fd} \cdot \tau_{fd} \right) = 0, \\ |F_f| \frac{\partial}{\partial t} (\boldsymbol{D}_f \cdot \boldsymbol{n}_f) - (H_e - H_d) = -|F_f| \boldsymbol{j}_f \cdot \boldsymbol{n}_f, \\ |\Phi_{fp}| \frac{\partial}{\partial t} (\boldsymbol{D}_{fp} \cdot \boldsymbol{v}_{fp}) - (H_p - H_f) = -|\Phi_{fp}| \boldsymbol{j}_f \cdot \boldsymbol{v}_{fp}, \\ |\Phi_{fq}| \frac{\partial}{\partial t} (\boldsymbol{D}_{fq} \cdot \boldsymbol{v}_{fq}) - (H_f - H_q) = -|\Phi_{fq}| \boldsymbol{j}_f \cdot \boldsymbol{v}_{fq}, \\ \sum_{f \in \partial P_p} |F_f| \boldsymbol{D}_f \cdot \boldsymbol{n}_f = \|P_p\|\rho_p, \\ \sum_{f \in \partial \Pi_d} (|\Phi_{fp}| \boldsymbol{D}_{fp} \cdot \boldsymbol{v}_{fp} + |\Phi_{fq}| \boldsymbol{D}_{fq} \cdot \boldsymbol{v}_{fq}) = \|\Pi_d\|\rho_d. \end{cases}$$
(3)

As for the initial system the discretized Gauss laws are satisfied at any time, provided they are satisfied at the initial time and if the following discretized charge conservation laws hold:

$$\|P_p\|\frac{\partial\rho_p}{\partial t} + \sum_{f\in\partial P_p} |F_f|\boldsymbol{j}_f \cdot \boldsymbol{n}_f = 0 \quad \text{and} \quad \|\Pi_d\|\frac{\partial\rho_d}{\partial t} + \sum_{f\in\partial\Pi_d} (|\boldsymbol{\Phi}_{fp}|\boldsymbol{j}_f \cdot \boldsymbol{\nu}_{fp} + |\boldsymbol{\Phi}_{fq}|\boldsymbol{j}_f \cdot \boldsymbol{\nu}_{fq}) = 0.$$

Otherwise the electric field must be corrected by various methods in order to enforce the Gauss law to be satisfied at any time [8,4,1-3].

Now notice that:

$$t_{f} = \frac{1}{\cos\theta_{fp}} \boldsymbol{v}_{fp} + \tan\theta_{fp} \boldsymbol{n}_{f} = \frac{1}{\cos\theta_{fq}} \boldsymbol{v}_{fq} + \tan\theta_{fq} \boldsymbol{n}_{f},$$

$$\tau_{fp} = \frac{1}{\cos\theta_{fp}} \boldsymbol{n}_{f} + \tan\theta_{fp} \boldsymbol{v}_{fp}, \quad \tau_{fq} = \frac{1}{\cos\theta_{fq}} \boldsymbol{n}_{f} + \tan\theta_{fq} \boldsymbol{v}_{fq},$$
(4)

and introduce the degrees of freedom $X_f = |F_f| E_f \cdot t_f$ and $E_f = |\Phi_{fp}| E_{fp} \cdot \tau_{fp} + |\Phi_{fq}| E_{fq} \cdot \tau_{fq}$. From (2), (3), (4) H_f can be calculated as a function of H_p , H_q , H_d , H_e and we obtain the following explicit differential system:

$$\begin{split} \left\{ \begin{array}{l} \frac{\partial H_p}{\partial t} + \frac{1}{\mu_p \|P_p\|} \sum_{f \in \partial P_p} X_f = 0, \\ \frac{\partial H_d}{\partial t} - \frac{1}{\mu_d \|\Pi_d\|} \sum_{f \in \partial \Pi_d} \mathcal{E}_f = 0, \\ \frac{\partial X_f}{\partial t} + \frac{|F_f|}{C_f^{\varepsilon}} (H_q - H_p) - \frac{S_f}{C_f^{\varepsilon}} (H_e - H_d) = -|F_f| \frac{C_f}{C_f^{\varepsilon}} \boldsymbol{j}_f \cdot \boldsymbol{t}_f, \\ \frac{\partial \mathcal{E}_f}{\partial t} - \frac{Q_f}{|F_f| C_f^{\varepsilon}} (H_e - H_d) + \frac{S_f}{C_f^{\varepsilon}} (H_q - H_p) = -\frac{|\Phi_{fp}|}{\varepsilon_p} \boldsymbol{j}_f \cdot \boldsymbol{\tau}_{fp} - \frac{|\Phi_{fq}|}{\varepsilon_q} \boldsymbol{j}_f \cdot \boldsymbol{\tau}_{fq} + \frac{R_f}{C_f^{\varepsilon}} \boldsymbol{j}_f \cdot \boldsymbol{t}_f, \end{split}$$

where μ_d , S_f , C_f , C_f^{ε} , Q_f and R_f are values depending on ε_p , μ_p and geometrical coefficients. This system can be solved by the leap-frog scheme for which the (experimental) stability condition is: $\Delta t \leq 1$ $\frac{1}{2}(\varepsilon\mu)^{1/2}\min(\min_p(\min_{f\in\partial P_p}|F_f|),\min_d(\min_{f\in\partial \Pi_d}(|\Phi_{fp}|+|\Phi_{fq}|))).$

896

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Fig. 2. Left: meshes. Right: isovalues of B.

3. Concluding remarks

The numerical experiments show the efficiency of the method even if arbitrary distorted meshes are used. For example Fig. 2 displays the isovalues of the magnetic field in the case of the calculation of the eigenmodes of a square cavity for which: $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ if $x \le 0.5$ and $\varepsilon = 2\varepsilon_0$, $\mu = \frac{1}{2}\mu_0$ if x > 0.5.

The results are similar to those obtained with the standard finite difference scheme [9].

The method coincides with the standard finite difference scheme [9] when the primal polygons are rectangles and with the finite volumes methods described in [4] when the primal (dual) mesh is a Delaunay (Voronoi) mesh.

Obviously this type of method can be applied to similar partial differential equations like the acoustic equations. It also can be generalized to anisotropic media.

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