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Partial Differential Equations

On a minimization problem related to lifting of BV functions with values in S^1

Arkady Poliakovsky

Department of Mathematics, Technion – I.I.T., 32000 Haifa, Israel

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Abstract

For $u \in W^{1,1}(\Omega, S^1)$ denote by K the set of minimizers of the problem $\min \int_\Omega |u \wedge \nabla u - D\phi|$, over $\phi \in \mathrm{BV}(\Omega)$ satisfying $\int_\Omega \phi = 0$. We show that an extreme point of K must be a lifting of u, up to an additive constant. We also prove a more general result for the case of u in $\mathrm{BV}(\Omega, S^1)$. ***To cite this article: A. Poliakovsky, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Sur un problème de minimisation lié au relèvement des fonctions BV à valeurs dans S^1. Pour $u \in W^{1,1}(\Omega, S^1)$ on désigne par K l'ensemble des minimiseurs pour le problème $\min \int_\Omega |u \wedge \nabla u - D\phi|$ sur l'ensemble des fonctions $\phi \in \mathrm{BV}(\Omega)$ vérifiant $\int_\Omega \phi = 0$. On démontre que chaque point extrême de K est un relèvement de u, à une constante additive près. On démontre ainsi une généralisation pour le cas $u \in \mathrm{BV}(\Omega, S^1)$. ***Pour citer cet article : A. Poliakovsky, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soit Ω un domaine borné de $\mathbb{R}^N$ ou le bord d'un domaine borné régulier $G \subset \mathbb{R}^{N+1}$ et soit $u \in W^{1,1}(\Omega, S^1)$. Brezis, Mironescu et Ponce ont démontré dans [2] que le problème de minimisation

$$\min\left\{\|D\varphi - u \wedge \nabla u\|_{\mathcal{M}(\Omega)} \colon \varphi \in \mathrm{BV}(\Omega)\right\}, \tag{1}$$

a au moins un minimiseur φ qui est un *relèvement* de u, c.à.d. $u = \mathrm{e}^{\mathrm{i}\varphi}$, p.p. dans Ω. Soit K l'ensemble de tous les minimiseurs dans (1) qui vérifient en plus la contrainte $\int_\Omega \varphi = 0$. La question suivante a été posée dans [2] (pour le cas $N = 2$) :

E-mail address: maarkady@techunix.technion.ac.il (A. Poliakovsky).

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Question. Est-ce qu'une fonction $\varphi \in K$ est un point extrême de K si et seulement si $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$ pour une constante $C \in \mathbb{R}$?

Un exemple simple montre qu'une fonction $\varphi \in K$ qui vérifie $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$ n'est pas forcément un point extrême de K. Mais on a le résultat positif suivant, valable pour tout N.

Théorème 0.1. *Pour $u \in W^{1,1}(\Omega, S^1)$, tout point extrême φ de K vérifie $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$, pour une constante C.*

En fait, Théorème 0.1 est un cas particulier d'un résultat plus général, valable pour $u \in \mathrm{BV}(\Omega, S^1)$. Le gradient d'un tel u se décompose comme suit (voir [1]) :

$$Du = D^a u + D^c u + D^j u, \quad \text{avec } D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner S(u),$$

où D^a, D^c et D^j sont respectivement les parties absolument continue, de Cantor et de saut de Du. L'ensemble de saut est désigné par $S(u)$. Ignat [5] a défini une mesure de Radon à valeurs dans $\mathbb{R}^N$ par

$$\mu = u \wedge D^a u + u \wedge D^c u + \rho(u^+, u^-) \nu_u \mathcal{H}^{N-1} \llcorner S(u),$$

où $\rho(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ est donnée par

$$\rho(w_1, w_2) = \begin{cases} \mathrm{Arg}(\frac{w_1}{w_2}) & \text{if } \frac{w_1}{w_2} \neq -1, \\ \mathrm{Arg}(w_1) - \mathrm{Arg}(w_2) & \text{if } \frac{w_1}{w_2} = -1, \end{cases} \quad \forall w_1, w_2 \in S^1.$$

La fonction Arg est définie par $\mathrm{Arg}(w) = \theta \in (-\pi, \pi]$ si $w = \mathrm{e}^{\mathrm{i}\theta}$. La généralisation suivante de la fonctionnelle de (1) pour le cas $u \in \mathrm{BV}(\Omega, S^1)$ est suggérée dans [5] :

$$J(\varphi) = \|D\varphi - \mu\|_{\mathcal{M}(\Omega)}, \quad \forall \varphi \in \mathrm{BV}(\Omega).$$

Comme avant on note K l'ensemble des minimiseurs de J, sous la contrainte $\int_\Omega \varphi = 0$. Notre généralisation du Théorème 0.1 est le résultat suivant.

Théorème 0.1′. *Pour $u \in \mathrm{BV}(\Omega, S^1)$, tout point extrême φ de K vérifie $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$, pour une constante C.*

1. Introduction and notation

Let Ω denote either a bounded domain in $\mathbb{R}^N$, or the boundary of a smooth bounded domain $G \subset \mathbb{R}^{N+1}$. An *optimal lifting* of a function $u = (u_1, u_2) \in W^{1,1}(\Omega, S^1)$ is a function φ realizing the minimum for

$$E(u) := \min\left\{ \int_\Omega |D\varphi| \colon \varphi \in \mathrm{BV}(\Omega) \text{ such that } u = \mathrm{e}^{\mathrm{i}\varphi},\ \mathcal{H}^N\text{-a.e. in } \Omega \right\}. \tag{2}$$

When Ω is a simply connected surface, the boundary of a smooth bounded domain in $\mathbb{R}^3$, it was shown by Brezis, Mironescu and Ponce [2] that

$$E(u) - \int_\Omega |\nabla u| = 2\pi L(u), \tag{3}$$

where $L(u)$ is the length of a minimal connection connecting the topological singularities of u (see [2] for details). It was shown in [2] (again when $N = 2$) that $L(u)$ is related also to another minimization problem, namely

$$2\pi L(u) = \min\left\{ \|D\varphi - u \wedge \nabla u\|_{\mathcal{M}(\Omega)} \colon \varphi \in \mathrm{BV}(\Omega) \right\}, \tag{4}$$

where

$$u \wedge \nabla u = \{u \wedge u_{x_i}\}_{i=1}^N = \{u_1(u_2)_{x_i} - u_2(u_1)_{x_i}\}_{i=1}^N,$$

and the total variation of the $\mathbb{R}^N$-valued Borel measure $\mu = (\mu_1, \dots, \mu_N)$ on Ω is given by

$$\|\mu\|_{\mathcal{M}(\Omega)} = \int_\Omega |\mathrm{d}\mu| = \sup\left\{\sum_{i=1}^N \int_\Omega \eta_i \,\mathrm{d}\mu_i \colon \eta_1, \dots, \eta_N \in C_c(\Omega) \text{ s.t. } \sum_{i=1}^N \eta_i^2 \leqslant 1 \text{ in } \Omega\right\}.$$

Following [2] we denote by K the set of minimizers in (4) satisfying $\int_\Omega \varphi = 0$. In this note we address the following question that was raised in [2] (for the case $N = 2$):

Question. Is it true that $\varphi \in K$ is an extreme point of K if and only if $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$ for some constant C?

A simple example (see below) shows that a function $\varphi \in K$ satisfying $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$ may not be an extreme point of K. On the other hand, we do have the following result (valid for any N):

Theorem 1.1. *For $u \in W^{1,1}(\Omega, S^1)$ every extreme point φ of K satisfies $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$, for some constant C.*

Actually, Theorem 1.1 will follow from a more general result which treats the case of $u \in \mathrm{BV}(\Omega, S^1)$, where the functional on the r.h.s. on (4) is replaced by a more general functional, as introduced by Ignat [5]. For such u we have (see [1]):

$$Du = D^a u + D^c u + D^j u, \quad \text{with } D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1} \llcorner S(u),$$

where D^a, D^c and D^j are the absolutely continuous, Cantor and jump part of Du, respectively. $S(u)$ represents the set of jump points of u which is a countably $\mathcal{H}^{N-1}$-rectifiable set on Ω oriented by the Borel map $\nu_u : S(u) \to S^{N-1}$. The Borel functions $u^+, u^- : S(u) \to S^1$ are the traces of u on the jump set $S(u)$ with respect to the orientation ν_u. Following [5] we define a finite Radon $\mathbb{R}^N$-valued measure μ by

$$\mu = u \wedge D^a u + \tilde{u} \wedge D^c u + \rho(u^+, u^-) \nu_u \mathcal{H}^{N-1} \llcorner S(u).$$

Here $\tilde{u}$ is the approximate limit of u that is defined $\mathcal{H}^{N-1}$-a.e. on $\Omega \setminus S_u$ and the function $\rho(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ is defined by

$$\rho(w_1, w_2) = \begin{cases} \mathrm{Arg}\left(\frac{w_1}{w_2}\right) & \text{if } \frac{w_1}{w_2} \neq -1, \\ \mathrm{Arg}(w_1) - \mathrm{Arg}(w_2) & \text{if } \frac{w_1}{w_2} = -1, \end{cases} \quad \forall w_1, w_2 \in S^1,$$

where $\mathrm{Arg}(w) = \theta \in (-\pi, \pi]$ for $w = \mathrm{e}^{\mathrm{i}\theta}$. Notice that the function ρ is antisymmetric, i.e.,

$$\rho(w_1, w_2) = -\rho(w_2, w_1), \quad \forall w_1, w_2 \in S^1,$$

and therefore μ does not depend on the choice of the orientation ν_u on the jump set $S(u)$. Following [5] we define the functional

$$J(\varphi) = \|D\varphi - \mu\|_{\mathcal{M}(\Omega)}, \quad \forall \varphi \in \mathrm{BV}(\Omega).$$

Note that J coincides with the functional on the r.h.s. of (4) when $u \in W^{1,1}(\Omega, S^1)$. As before we denote by K the set of minimizers of J, under the constraint $\int_\Omega \varphi = 0$. We shall prove the following generalization of Theorem 1.1.

Theorem 1.1′. *For $u \in \mathrm{BV}(\Omega, S^1)$ every extreme point φ of K satisfies $u = \mathrm{e}^{\mathrm{i}(\varphi + C)}$, for some constant C.*

2. Main proposition

Theorem 1.1′ is an easy consequence of the next proposition:

Proposition 2.1. *Let $\varphi \in \mathrm{BV}(\Omega)$ be such that the function $\mathrm{e}^{-\mathrm{i}\varphi}u$ is not equal to a constant, $\mathcal{H}^N$-a.e. in Ω. Then there exist $\varphi_1, \varphi_2 \in \mathrm{BV}(\Omega, \mathbb{R})$ with $D\varphi_1 \neq D\varphi_2$ (i.e., $|D\varphi_1 - D\varphi_2|_{\mathrm{BV}} = \|D\varphi_1 - D\varphi_2\|_{\mathcal{M}(\Omega)} > 0$) such that $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$ and $J(\varphi) = \frac{1}{2}J(\varphi_1) + \frac{1}{2}J(\varphi_2)$.*

Proof. It is well known (see [4,3]) that there exists $\varphi_0 \in \mathrm{BV}(\Omega)$ such that $u = \mathrm{e}^{\mathrm{i}\varphi_0}$, $\mathcal{H}^N$-a.e. in Ω. By Volpert's chain rule for BV functions (see [1]) we have

$$D^a\varphi_0 = u \wedge D^a u \quad \text{and} \quad D^c\varphi_0 = \tilde{u} \wedge D^c u. \tag{5}$$

Set $f_1(s) = \cos s$ and $f_2(s) = \sin s$. The functions f_1, f_2 are 2π-periodic and satisfy $|f_i'(s)| \leqslant 1$ for all s, $i = 1, 2$, so in particular,

$$\bigl|f_i(s) - f_i(t)\bigr| \leqslant |s - t| \quad \forall s, t \in \mathbb{R},\ i = 1, 2. \tag{6}$$

Define

$$v_i(x) = f_i\bigl(\varphi(x) - \varphi_0(x)\bigr) \quad \forall x \in \Omega,\ i = 1, 2.$$

Again by the chain rule we have that $v_i \in \mathrm{BV}(\Omega)$, $i = 1, 2$, and using also (5) we get:

$$\begin{aligned} D^a v_i &= f_i'(\varphi - \varphi_0)(D^a\varphi - D^a\varphi_0) = f_i'(\varphi - \varphi_0)(D^a\varphi - u \wedge D^a u),\\ D^c v_i &= f_i'(\tilde{\varphi} - \tilde{\varphi}_0)(D^c\varphi - D^c\varphi_0) = f_i'(\tilde{\varphi} - \tilde{\varphi}_0)(D^c\varphi - \tilde{u} \wedge D^c u). \end{aligned} \tag{7}$$

The jump part $D^j v_i$ is concentrated on the set $S(\varphi) \cup S(\varphi_0)$. We have:

$$\begin{aligned} &\varphi_0^+ - \varphi_0^- \in 2\pi\mathbb{Z} \quad \mathcal{H}^{N-1}\text{-a.e.} \quad \text{on } S(\varphi_0) \setminus S(u),\\ &\varphi_0^+ - \varphi_0^- - \rho(u^+, u^-) \in 2\pi\mathbb{Z} \quad \text{on } S(u). \end{aligned}$$

The orientation on $S(\varphi_0)$, $S(\varphi)$, and $S(u)$ is chosen to be the same on any intersection of two of these three sets. Denote this orientation vector by ν. Since f_i is 2π-periodic, we obtain that

$$D^j v_i = \bigl[f_i\bigl(\varphi^+ - \rho(u^+, u^-) - \varphi_0^-\bigr) - f_i(\varphi^- - \varphi_0^-)\bigr]\nu\mathcal{H}^{N-1} \llcorner \bigl(S(\varphi) \cup S(u)\bigr). \tag{8}$$

Note that our assumption $\mathrm{e}^{-\mathrm{i}\varphi}u \not\equiv \mathit{const}$ implies that at least one of v_1 and v_2 is not equal to a constant, $\mathcal{H}^N$-a.e. in Ω.

Without loss of generality we may assume then that $|Dv_1|_{\mathrm{BV}(\Omega)} > 0$. Set $\varphi_1 = \varphi + v_1$ and $\varphi_2 = \varphi - v_1$. Obviously, $|D\varphi_1 - D\varphi_2|_{\mathrm{BV}(\Omega)} > 0$ and $\varphi = \frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$.

Since the absolutely continuous, the Cantor and the jump parts of measures are mutually singular, we have for any $\psi \in \mathrm{BV}(\Omega)$:

$$\begin{aligned} J(\psi) = {} & \int_\Omega |D^a\psi - u \wedge D^a u|\,\mathrm{d}\mathcal{H}^N + \|D^c\psi - \tilde{u} \wedge D^c u\|_{\mathcal{M}(\Omega)}\\ & + \int_{S(\psi)\cup S(u)} \bigl|\psi^+ - \psi^- - \rho(u^+, u^-)\bigr|\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned} \tag{9}$$

Here the orientation of ν_ψ is chosen to be equal to that of ν_u on $S(\psi) \cap S(u)$. Applying (9) to φ_1, φ_2, using (7) and (8), yields

$$J(\varphi_1)=\int_\Omega \left|1+f_1'(\varphi-\varphi_0)\right||D^a\varphi-u\wedge D^a u|+\int_\Omega \left|1+f_1'(\tilde\varphi-\tilde\varphi_0)\right|\mathrm{d}\big(|D^c\varphi-u\wedge D^c u|\big)$$
$$+\int_{S(\varphi)\cup S(u)}\left|\left[\varphi^+-\varphi^--\rho(u^+,u^-)\right]+\left[f_1\big(\varphi^+-\rho(u^+,u^-)-\varphi_0^-\big)-f_1(\varphi^--\varphi_0^-)\right]\right|\mathrm{d}\mathcal{H}^{N-1},\tag{10}$$

and

$$J(\varphi_2)=\int_\Omega \left|1-f_1'(\varphi-\varphi_0)\right||D^a\varphi-u\wedge D^a u|+\int_\Omega \left|1-f_1'(\tilde\varphi-\tilde\varphi_0)\right|\mathrm{d}\big(|D^c\varphi-u\wedge D^c u|\big)$$
$$+\int_{S(\varphi)\cup S(u)}\left|\left[\varphi^+-\varphi^--\rho(u^+,u^-)\right]-\left[f_1\big(\varphi^+-\rho(u^+,u^-)-\varphi_0^-\big)-f_1(\varphi^--\varphi_0^-)\right]\right|\mathrm{d}\mathcal{H}^{N-1}.\tag{11}$$

Since $|f_1'|\leqslant 1$ we have

$$\left|1\pm f_1'(\varphi-\varphi_0)\right|=1\pm f_1'(\varphi-\varphi_0)\quad\text{and}\quad\left|1\pm f_1'(\tilde\varphi-\tilde\varphi_0)\right|=1\pm f_1'(\tilde\varphi-\tilde\varphi_0).$$

It follows that

$$\int_\Omega \left|1\pm f_1'(\varphi-\varphi_0)\right||D^a\varphi-u\wedge D^a u|=\int_\Omega \big(1\pm f_1'(\varphi-\varphi_0)\big)|D^a\varphi-u\wedge D^a u|\tag{12}$$

and

$$\int_\Omega \left|1\pm f_1'(\tilde\varphi-\tilde\varphi_0)\right|\mathrm{d}\big(|D^c\varphi-u\wedge D^c u|\big)=\int_\Omega \big(1\pm f_1'(\tilde\varphi-\tilde\varphi_0)\big)\,\mathrm{d}\big(|D^c\varphi-u\wedge D^c u|\big).\tag{13}$$

Furthermore, by (6) we have

$$\left|f_1\big(\varphi^+-\rho(u^+,u^-)-\varphi_0^-\big)-f_1(\varphi^--\varphi_0^-)\right|\leqslant\left|\varphi^+-\varphi^--\rho(u^+,u^-)\right|.$$

Therefore, denoting $\sigma(x)=\operatorname{sgn}(\varphi^+(x)-\varphi^-(x)-\rho(u^+(x),u^-(x)))$ for $x\in S(\varphi)\cup S(u)$, we have

$$\int_{S(\varphi)\cup S(u)}\left|\left[\varphi^+-\varphi^--\rho(u^+,u^-)\right]\pm\left[f_1\big(\varphi^+-\rho(u^+,u^-)-\varphi_0^-\big)-f_1(\varphi^--\varphi_0^-)\right]\right|\mathrm{d}\mathcal{H}^{N-1}$$
$$=\int_{S(\varphi)\cup S(u)}\sigma\cdot\Big(\left[\varphi^+-\varphi^--\rho(u^+,u^-)\right]\pm\left[f_1\big(\varphi^+-\rho(u^+,u^-)-\varphi_0^-\big)-f_1(\varphi^--\varphi_0^-)\right]\Big)\,\mathrm{d}\mathcal{H}^{N-1}.\tag{14}$$

Adding together (10) and (11), taking into account (12)–(14), gives $J(\varphi)=\frac{1}{2}J(\varphi_1)+\frac{1}{2}J(\varphi_2)$. □

3. Proof of Theorem 1.1′

Proof. Let φ be an extreme point of K and assume by contradiction that $u\,\mathrm{e}^{-\mathrm{i}\varphi}\not\equiv const$. Therefore, by Proposition 2.1, there exist $\varphi_1,\varphi_2\in \mathrm{BV}(\Omega)$ such that $\varphi=\frac{1}{2}\varphi_1+\frac{1}{2}\varphi_2$, $J(\varphi)=\frac{1}{2}J(\varphi_1)+\frac{1}{2}J(\varphi_2)$ with $|D\varphi_1-D\varphi_2|_{\mathrm{BV}(\Omega)}>0$. Defining

$$\bar\varphi_i(x)=\varphi_i(x)-\frac{\int_\Omega\varphi_i}{\mathcal{H}^N(\Omega)},\quad i=1,2,$$

we find easily that $\int_\Omega\bar\varphi_i=0$, $i=1,2$, $\varphi=\frac{1}{2}\bar\varphi_1+\frac{1}{2}\bar\varphi_2$ and $J(\varphi)=\frac{1}{2}J(\bar\varphi_1)+\frac{1}{2}J(\bar\varphi_2)$. Since φ is a minimizer, $\bar\varphi_1$ and $\bar\varphi_2$ are also minimizers, i.e., they belong to K. Contradiction. □

An immediate consequence is an alternative simple proof of the following result from [2] (for $u \in W^{1,1}(\Omega, S^1)$) and [5] (for $u \in \mathrm{BV}(\Omega, S^1)$) which holds for arbitrary N.

Corollary 3.1. *For any $u \in \mathrm{BV}(\Omega, S^1)$ there is at least one minimizer $\psi \in \mathrm{BV}(\Omega)$ of J which is a lifting of u, i.e., $u = \mathrm{e}^{\mathrm{i}\psi}$ a.e. in Ω.*

Proof. Since K is convex and compact in $L^1(\Omega)$ it follows from the Krein–Milman theorem that it has an extreme point φ. By Theorem 1.1′, $u = \mathrm{e}^{\mathrm{i}(\varphi+C)}$ so that $\psi = \varphi + C$ is a lifting which is also a minimizer for J. □

The following example shows that the converse to Theorem 1.1 and Theorem 1.1′ is false (in general).

Example. Let Ω be the unit disc in $\mathbb{R}^2 = \mathbb{C}$ and consider $u(z) = (z/|z|)^2 = \mathrm{e}^{2\theta\mathrm{i}} \in W^{1,1}(\Omega)$. Set

$$\varphi_1(z) = 2\,\mathrm{Arg}(\mathrm{e}^{\mathrm{i}\theta}) = 2\theta \quad \text{and} \quad \varphi_2(z) = 2\,\mathrm{Arg}(-\mathrm{e}^{\mathrm{i}\theta}).$$

Then both φ_1 and φ_2 belong to $\mathrm{BV}(\Omega)$ and satisfy $\mathrm{e}^{\mathrm{i}\varphi_1} = \mathrm{e}^{\mathrm{i}\varphi_2} = u$ and $J(\varphi_1) = J(\varphi_2) = 4\pi$. By the same argument as in [3, Lemma 4.1] we get that $E(u) = 2E(\mathrm{e}^{\mathrm{i}\theta}) = 8\pi$ (see (2)) and since $\int_\Omega |\nabla u| = 2\int_\Omega |\nabla\theta| = 4\pi$ we deduce from (3)–(4) that

$$\min\big\{\|D\varphi - u \wedge \nabla u\|_{\mathcal{M}(\Omega)}\colon\ \varphi \in \mathrm{BV}(\Omega)\big\} = 4\pi.$$

Put $\bar\varphi_i = \varphi_i - \int_\Omega \varphi_i/|\Omega|$, $i = 1, 2$, and $\varphi_0(x) = \frac{1}{2}\bar\varphi_1(x) + \frac{1}{2}\bar\varphi_2(x)$. Note that $\varphi_0 \in K$ and $u = \mathrm{e}^{\mathrm{i}(\varphi_0+C)}$ for some C, but clearly φ_0 is not an extreme point of K.

The question in [2] was originally raised in the case where Ω is a surface. Let

$$\Omega = S^2 = \big\{(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)\colon\ -\pi < \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant \pi\big\},$$

and $u = \mathrm{e}^{2\theta\mathrm{i}}$ on Ω. A straightforward modification of the above argument yields $\varphi_0 \in K$ such that $u = \mathrm{e}^{\mathrm{i}(\varphi_0+C)}$ which is not an extreme point of K.

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