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Risk premium and fair option prices under stochastic volatility: the HARA solution

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Abstract

We have solved the problem of finding (HARA) fair option price under a general stochastic volatility model. For a given HARA utility, the 'risk premium', i.e., the 'market price of volatility risk' is determined via a solution of a certain nonlinear PDE. Equivalently, the fair option price is determined as a solution of an uncoupled system of a non-linear PDE and a Black–Scholes type PDE. ***To cite this article: S.D. Stojanovic, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Prime de risque et prix de l'option sous l'hypothèse de volatilité stochastique. Nous résolvons le problème de la détermination du prix de l'option sous un modèle général de volatilité stochastique. Pour une fonction d'utilité HARA, « la prime de risque », i.e. le prix du marché associé au risque de volatilité, est déterminé en utilisant la solution d'une EDP non linéaire. De la même façon, le prix de l'option est déterminé comme solution d'un système découplé d'une EDP non linéaire et d'une EDP de type Black–Scholes. ***Pour citer cet article : S.D. Stojanovic, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Considérons un « marché boursier à volatilité stochastique », consistant d'un titre unique au prix $Y(t)$ à la date t et vérifiant le système suivant d'équations différentielles stochastiques de type Itô

$$\mathrm{d}Y(t) = Y(t)\big(\mathbb{c}\big(t, Y(t), \mathbb{v}(t)\big) - \mathbb{D}_1\big(t, Y(t), \mathbb{v}(t)\big)\big)\,\mathrm{d}t + Y(t)\,p\big(t, Y(t), \mathbb{v}(t)\big)\,\mathrm{d}B_1(t) \tag{1}$$

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$$\mathrm{d}\mathbb{v}(t)=q\big(t,Y(t),\mathbb{v}(t)\big)\,\mathrm{d}t+w\big(t,Y(t),\mathbb{v}(t)\big)\Big(\rho\big(t,Y(t),\mathbb{v}(t)\big)\,\mathrm{d}B_1(t)+\sqrt{1-\rho\big(t,Y(t),\mathbb{v}(t)\big)^2}\,\mathrm{d}B_2(t)\Big)$$

où B_1 et B_2 représentent des mouvements browniens indépendants, $\mathbb{a}$ est le taux d'appréciation du titre, $\mathbb{D}_1$ est le dividende, p est la volatilité, et v est un facteur scalaire arbitraire (par exemple, la volatilité : $p(t,Y,\mathbb{v})=\mathbb{v}$) à direction q, à diffusion w et où $-1\leqslant\rho\leqslant 1$ est le coefficient de corrélation entre prix et facteur. Soit r le taux d'intérêt.

Soit également un marché à options constitué d'une option unique sur le titre ci-dessus, de gain fixe $\upsilon(Y)$ à la date de maturité T, et au prix $V(t,Y(t),\mathbb{v}(t))$ à la date $t<T$, et où $V(t,Y,\mathbb{v})$ est une fonction inconnue a priori.

Soit $X>0$, la richesse. L'utilité associée à la richesse est mesurée par une fonction d'utilité de type HARA : $\psi_\gamma(X)=X^{1-\gamma}/(1-\gamma)$ pour $\gamma\in(0,\infty)$, $\gamma\neq 1$, et $\psi_1(X)=\log(X)$.

Considérons une stratégie de couverture du portefeuille par autofinancement $\Pi(t,X,Y,\mathbb{v})=\{\Pi_1,\Pi_2\}$, où Π_1 est la valeur monétaire de l'investissement dans l'action et Π_2 est la valeur de l'investissement dans l'option. Etant donné (1) et une stratégie fixe Π, il existe une EDS qui caractérise l'évolution de la richesse $X(t)=X^{\Pi}(t)$. La stratégie de portefeuille γ-optimale (le problème de Merton), notée $\Pi^\star_\gamma=\big\{\Pi^\star_{\gamma 1},\Pi^\star_{\gamma 2}\big\}$ est telle que :

$$\sup_{\Pi} E_{t,X,Y,\mathbb{v}}\psi_\gamma\big(X^{\Pi}(T)\big)=E_{t,X,Y,\mathbb{v}}\psi_\gamma\big(X^{\Pi^\star_\gamma}(T)\big). \tag{2}$$

Définition 0.1. Pour tout $\gamma\in(0,\infty)$, la fonction du prix de l'option $V_\gamma(t,Y,\mathbb{v})$ est définie comme la solution d'une équation Black–Scholes (abstraite) $\Pi^\star_{\gamma 2}=0$.

Theorem 0.2. *Pour $\gamma\in(0,\infty)$, $\gamma\neq 1$, le prix de l'option correspondant $V_\gamma(t,Y,\mathbb{v})$ est solution de l'EDP Black–Scholes généralisée*

$$\begin{aligned}&V_\gamma{}^{(1,0,0)}+\frac{1}{2}Y^2V_\gamma{}^{(0,2,0)}p^2+Yw\rho V_\gamma{}^{(0,1,1)}p+\frac{1}{2}w^2V_\gamma{}^{(0,0,2)}+Y(r-\mathbb{D}_1)V_\gamma{}^{(0,1,0)}-rV_\gamma\\&\quad+\left(q-\frac{(\mathbb{a}-r)w\rho}{p}+w^2(1-\rho^2)\frac{f_\gamma{}^{(0,0,1)}}{f_\gamma}\right)V_\gamma{}^{(0,0,1)}=0\end{aligned} \tag{3}$$

de condition terminale $V_\gamma(T,Y,\mathbb{v})=\upsilon(Y)$, et où f_γ est une solution appropriée de l'EDP de type Monge–Ampère

$$\begin{aligned}&-\left(\frac{(\mathbb{a}-r)^2}{2p^2}+r\gamma\right)f_\gamma{}^2+\left(\frac{q\gamma}{\gamma-1}-\frac{w(\mathbb{a}-r)\rho}{p}\right)f_\gamma{}^{(0,0,1)}f_\gamma+\frac{w^2\gamma}{2(\gamma-1)}f_\gamma{}^{(0,0,2)}f_\gamma\\&\quad+\frac{Y(\mathbb{a}-r+\gamma(r-\mathbb{D}_1))}{\gamma-1}f_\gamma{}^{(0,1,0)}f_\gamma+\frac{pwY\gamma\rho}{\gamma-1}f_\gamma{}^{(0,1,1)}f_\gamma+\frac{p^2Y^2\gamma}{2(\gamma-1)}f_\gamma{}^{(0,2,0)}f_\gamma\\&\quad+\frac{\gamma}{\gamma-1}f_\gamma{}^{(1,0,0)}f_\gamma-\frac{1}{2}w^2\rho^2\big(f_\gamma{}^{(0,0,1)}\big)^2-\frac{1}{2}p^2Y^2\big(f_\gamma{}^{(0,1,0)}\big)^2-pwY\rho f_\gamma{}^{(0,0,1)}f_\gamma{}^{(0,1,0)}=0\end{aligned} \tag{4}$$

de condition terminale $f_\gamma(T,Y,\mathbb{v})=1$.

Pour $\gamma=1$, le prix de l'option correspondant $V_1=V_1(t,Y,\mathbb{v})$ est solution de l'EDP Black–Scholes généralisée (3) *où $f_1=1$, soit pour $f_1{}^{(0,0,1)}/f_1=0$* ((4) *n'est pas nécessaire dans ce cas*).

1. Statement of problem and it's solution

Consider a 'stochastic volatility stock market' (cf. [1,3,4,6–8,11]), consisting of a single security with price $Y(t)$ at time t, obeying Itô SDE system

$$\begin{aligned}&\mathrm{d}Y(t)=Y(t)\big(\mathbb{a}\big(t,Y(t),\mathbb{v}(t)\big)-\mathbb{D}_1\big(t,Y(t),\mathbb{v}(t)\big)\big)\,\mathrm{d}t+Y(t)p\big(t,Y(t),\mathbb{v}(t)\big)\,\mathrm{d}B_1(t)\\&\mathrm{d}\mathbb{v}(t)=q\big(t,Y(t),\mathbb{v}(t)\big)\mathrm{d}t+w\big(t,Y(t),\mathbb{v}(t)\big)\Big(\rho\big(t,Y(t),\mathbb{v}(t)\big)\,\mathrm{d}B_1(t)+\sqrt{1-\rho\big(t,Y(t),\mathbb{v}(t)\big)^2}\,\mathrm{d}B_2(t)\Big)\end{aligned} \tag{5}$$

where B_1 and B_2 are independent standard Brownian motions, and where $\mathbb{a}$ is the appreciation rate, $\mathbb{D}_1$ is the dividend rate, p is the volatility, and $\mathbb{v}$ is an arbitrary scalar factor (for example, volatility: $p(t, Y, \mathbb{v}) = \mathbb{v}$), with a drift q, diffusion w, and where $-1 \leqslant \rho \leqslant 1$ is the instantaneous price/factor correlation. Let r be the interest rate.

Consider also an associated option market, consisting of a single option on the above underlying, with a fixed payoff $\upsilon(Y)$ (for example, $\upsilon_{\text{call}}(Y) = \max[0, Y - k]$, for some strike price k) at a fixed expiration time T, with a price $V(t, Y(t), \mathbb{v}(t))$ at time $t < T$, where function $V(t, Y, \mathbb{v})$ is a priori unknown.

The problem is to characterize, and be able to compute, 'a fair option price function' $V(t, Y, \mathbb{v})$.

Let $X > 0$ denote the wealth. We measure utility of wealth via HARA class of utility functions $\psi_\gamma(X) = X^{1-\gamma}/(1-\gamma)$, for $\gamma \in (0, \infty)$, $\gamma \neq 1$, and $\psi_1(X) = \log(X)$. Parameter γ is called *risk-aversion*.

Consider a self-financing portfolio hedging strategy $\Pi(t, X, Y, \mathbb{v}) = \{\Pi_1(t, X, Y, \mathbb{v}), \Pi_2(t, X, Y, \mathbb{v})\}$, where Π_1 is the cash value of the investment into the underlying stock, and Π_2 is the cash value of the investment into the option. Assuming (5), and for a fixed strategy Π, it is not difficult to write down an Itô SDE characterizing corresponding evolution of the wealth $X(t) = X^{\Pi}(t)$. The γ-optimal portfolio strategy (Merton's problem; see [9,10]) is a strategy $\Pi^\star_\gamma = \{\Pi^\star_{\gamma 1}, \Pi^\star_{\gamma 2}\}$, such that

$$\sup_{\Pi} E_{t,X,Y,\mathbb{v}} \psi_\gamma\big(X^{\Pi}(T)\big) = E_{t,X,Y,\mathbb{v}} \psi_\gamma\big(X^{\Pi^\star_\gamma}(T)\big). \tag{6}$$

Definition 1.1. For any risk-aversion $\gamma \in (0, \infty)$, the fair option price function $V_\gamma(t, Y, \mathbb{v})$ is defined as a (smooth) solution of the (abstract) Black–Scholes equation $\Pi^\star_{\gamma 2} = 0$.

'Smooth' means once continuously differentiable in t, and twice continuously differentiable in variables Y and $\mathbb{v}$.

Intuitively, the fair option price is such a price for which it is not rational to speculate by investing, long or short, into options. Variants of this definition have been present in the literature already (see, e.g., [8]; cf. Davis' fair price definition – see, e.g., (7.3) and (7.4) in [7]). As it turns out, Definition 1.1, for each $\gamma \in (0, \infty)$ implies a selection of the 'risk premium' process (see (9) below). For some other issues in regard to the risk premium see, e.g., [11], where a notion of *viability* of the risk premium process is studied in details. We shall use alternative notations for partial derivatives: for example, $\partial^2 V(t, Y, \mathbb{v})/\partial \mathbb{v}^2 = V^{(0,0,2)}(t, Y, \mathbb{v})$.

Theorem 1.2. *For a risk-aversion $\gamma \in (0, \infty), \gamma \neq 1$, the corresponding (smooth) fair option price $V_\gamma = V_\gamma(t, Y, \mathbb{v})$ is a solution of a generalized Black–Scholes PDE*:

$$\begin{aligned} &{V_\gamma}^{(1,0,0)} + \frac{1}{2} Y^2 {V_\gamma}^{(0,2,0)} p^2 + Y w \rho {V_\gamma}^{(0,1,1)} p + \frac{1}{2} w^2 {V_\gamma}^{(0,0,2)} + Y(r - \mathbb{D}_1) {V_\gamma}^{(0,1,0)} - r V_\gamma \\ &\quad + \left(q - \frac{(\mathbb{a} - r) w \rho}{p} + w\sqrt{1-\rho^2}\,\mathbb{M}_\gamma\right) {V_\gamma}^{(0,0,1)} = 0 \end{aligned} \tag{7}$$

with the terminal condition

$$V_\gamma(T, Y, \mathbb{v}) = \upsilon(Y) \tag{8}$$

and with

$$\mathbb{M}_\gamma = w\sqrt{1-\rho^2}\,\frac{{f_\gamma}^{(0,0,1)}}{f_\gamma} \tag{9}$$

where $f_\gamma = f_\gamma(t, Y, \mathbb{v})$ is an appropriate solution of a Monge–Ampère type PDE (see, e.g. [2])

$$-\left(\frac{(\mathbb{a} - r)^2}{2p^2} + r\gamma\right) {f_\gamma}^2 + \left(\frac{q\gamma}{\gamma - 1} - \frac{w(\mathbb{a} - r)\rho}{p}\right) {f_\gamma}^{(0,0,1)} f_\gamma + \frac{w^2 \gamma}{2(\gamma - 1)} {f_\gamma}^{(0,0,2)} f_\gamma$$

$$+\frac{Y(\mathbb{a}-r+\gamma(r-\mathbb{D}_1))}{\gamma-1}{f_\gamma}^{(0,1,0)}f_\gamma+\frac{pwY\gamma\rho}{\gamma-1}{f_\gamma}^{(0,1,1)}f_\gamma+\frac{p^2Y^2\gamma}{2(\gamma-1)}{f_\gamma}^{(0,2,0)}f_\gamma$$
$$+\frac{\gamma}{\gamma-1}{f_\gamma}^{(1,0,0)}f_\gamma-\frac{1}{2}w^2\rho^2\left({f_\gamma}^{(0,0,1)}\right)^2-\frac{1}{2}p^2Y^2\left({f_\gamma}^{(0,1,0)}\right)^2-pwY\rho{f_\gamma}^{(0,0,1)}{f_\gamma}^{(0,1,0)}=0 \tag{10}$$

with the terminal condition:

$$f_\gamma(T,Y,\mathbb{v})=1. \tag{11}$$

For $\gamma=1$, i.e., in the case of log-*utility, the corresponding fair option price $V_1=V_1(t,Y,\mathbb{v})$ is characterized as a solution of the generalized Black–Scholes PDE* (7) *with* $\mathbb{M}_1=0$.

Remark 1. Comparing (7) with the literature (see, e.g., Eq. (15) in [4]) we can see that Theorem 1.2 implies that, in the case of HARA utility, for a risk-aversion $\gamma\in(0,\infty)$, the 'market price of volatility risk', or 'risk premium', is equal to $\mathbb{M}=\mathbb{M}_\gamma$, above characterized by (9)–(11).

Remark 2. It is probably quite useful to have an intuitive understanding of what the risk premium is. For example, after mimicking the classical derivation of the Black–Scholes PDE (see, e.g., Section 3.2.1 of [12]), in the case of the underlying price dynamics (5), one can see that pricing options under stochastic volatility is done as if, after the randomness is eliminated as much as possible (by means of hedging $\Pi^\star_\infty=\{-w\rho V^{(0,0,1)}/p-YV^{(0,1,0)},V\}$, which is a stochastic volatility analogue of the Black–Scholes hedging $\{-YV^{(0,1,0)},V\}$), the remaining white noise $\mathrm{d}B_2(t)/\mathrm{d}t$ is thought of as, or substituted by, the risk premium $\mathbb{M}$. This suggests that arbitrage arguments, if applied in the stochastic volatility setting, would require extreme considerations, such as $\mathbb{M}\to\pm\infty$ (cf. the notion of *lower and upper arbitrage prices* and Theorem 6.1 in [7]). We continue this discussion in Remark 7 below.

Remark 3. Analyzing (7), one can see that fair option price is unique, i.e., it is same for all $\gamma\in(0,\infty)$, iff $\mathbb{M}_\gamma=\mathbb{M}_1=0$, and assuming furthermore $w^2(1-\rho^2)>0$, iff ${f_\gamma}^{(0,0,1)}=0$. Furthermore, analyzing (10), one can conclude that (say, $r>0$) for ${f_\gamma}^{(0,0,1)}=0$ to hold it is *not* sufficient that the square of the Sharpe ratio $((\mathbb{a}-r)/p)^2$ does not depend on the factor $\mathbb{v}$, unless, additionally, which is trivial, the same holds for the volatility $p(t,Y,\mathbb{v})=p(t,Y)$.

Remark 4. Dividing by f_γ, (10) can be written also as a quasi-linear PDE. Further, multiplying by $(\gamma-1)$, and sending $\gamma\to1$, one arrives (at least formally) at

$$q{f_1}^{(0,0,1)}+\frac{w^2}{2}{f_1}^{(0,0,2)}+Y(\mathbb{a}-\mathbb{D}_1){f_1}^{(0,1,0)}+pwY\rho{f_1}^{(0,1,1)}+\frac{p^2Y^2}{2}{f_1}^{(0,2,0)}+{f_1}^{(1,0,0)}=0 \tag{12}$$

which (is a linear equation, and) together with (10), is solved, obviously, by $f_1=1$.

Remark 5. System (7)–(11) is *uncoupled*, which is very useful. Furthermore, f_γ and therefore also $\mathbb{M}_\gamma$ do not depend on a particular option payoff υ. Assuming that $\mathbb{M}_\gamma\neq0$ is continuous, since (11) implies that at the expiration time $\mathbb{M}_\gamma(T,Y,\mathbb{v})=0$, we conclude that even if data is time-independent, the (HARA) market price of volatility risk $\mathbb{M}_\gamma$ is not.

Remark 6. If $w=0$, or if $\rho=\pm1$, then $\mathbb{M}_\gamma=0$, and the same fair option price $V=V_\gamma=V_\gamma(t,Y,\mathbb{v})$ holds for all $\gamma\in(0,\infty)$. For example, if $w=0$, the price is characterized as a solution of a (hypoelliptic (see [5], and also [12,13])) Black–Scholes PDE

$$V^{(1,0,0)}+\frac{1}{2}Y^2V^{(0,2,0)}p^2+Y(r-\mathbb{D}_1)V^{(0,1,0)}-rV+qV^{(0,0,1)}=0. \tag{13}$$

If also $q=0$, then (13) simplifies further to the (usual) Black–Scholes PDE.

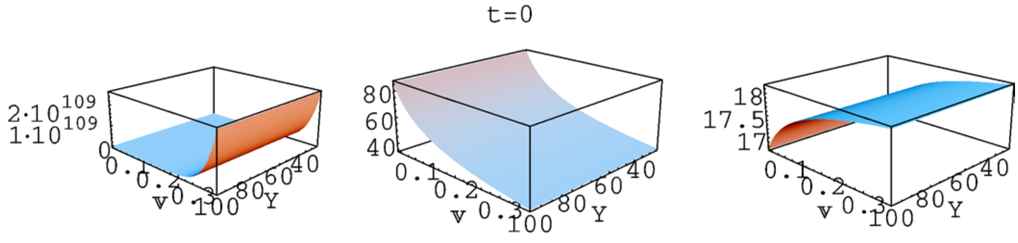

Fig. 1. f_{γ_1}, the corresponding $f_{\gamma_1}{}^{(0,0,1)}/f_{\gamma_1}$, and the market price of volatility risk $\mathbb{M}_{\gamma_1}$, for $\gamma_1 = 1/10$.

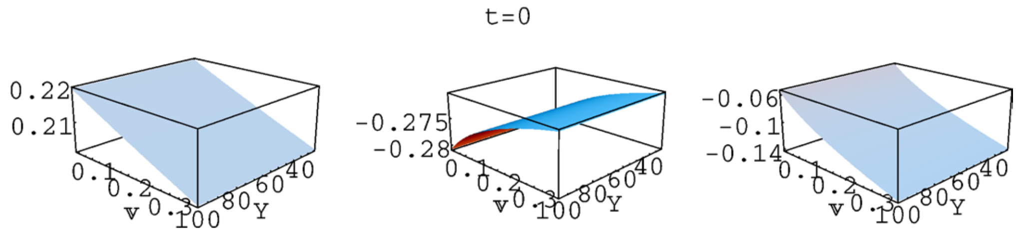

Fig. 2. As Fig. 1, but for $\gamma_2 = 100$.

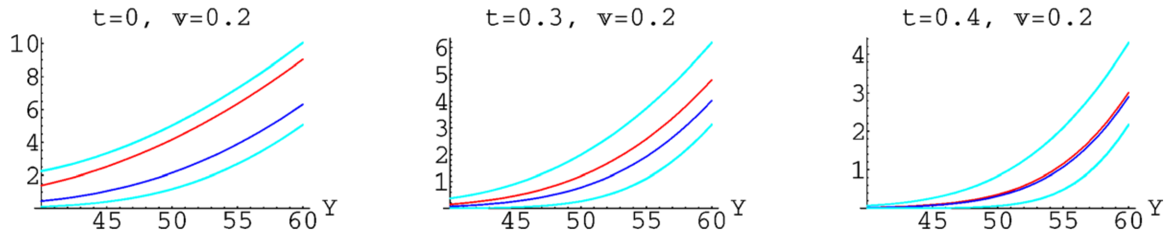

Fig. 3. Approximations of upper arbitrage, upper fair, lower fair, and lower arbitrage prices.

2. Computational example

Let $\mathbb{a} = (3\sqrt{\mathbb{v}} + 0.1)\sqrt{\mathbb{v}}$, $\mathbb{D}_1 = 0$, $p = \sqrt{\mathbb{v}}$, $q = 16(0.12 - \mathbb{v})$, $\rho = 1/2$, $w = \sqrt{\mathbb{v}}$, which amounts to a very volatile market, and let $r = 0.025$, $T = 0.5$.

For $\gamma_1 = 1/10$, and some time $t < T$, f_{γ_1}, the corresponding $f_{\gamma_1}{}^{(0,0,1)}/f_{\gamma_1}$, and the market price of volatility risk (the risk premium) $\mathbb{M}_{\gamma_1}$ are shown in Fig. 1, while for $\gamma_2 = 100$, and same time $t < T$, f_{γ_2}, the corresponding $f_{\gamma_2}{}^{(0,0,1)}/f_{\gamma_2}$, and the market price of volatility risk $\mathbb{M}_{\gamma_2}$ are shown in Fig. 2.

Remark 7. We continue Remark 2 above, and compare the methodology of HARA fair option prices of this paper, with the concept of lower and upper arbitrage prices (see [7]). We compute the fair option prices (for a call with strike $k = 60$) corresponding to γ_1 and γ_2, and refer to them as upper and lower fair prices. Furthermore, denote the solution of (7)–(8) corresponding to a particular risk premium $\mathbb{M}$ as $\mathbb{V}_{\mathbb{M}}$ (for example, $V_\gamma = \mathbb{V}_{\mathbb{M}_\gamma}$). We approximate the lower and upper arbitrage prices by $\mathbb{V}_{\pm\mathbb{m}}$, where $\mathbb{m}$ is sufficiently large. For example, if $\mathbb{m} = 100, 200$, in the above example we compute and plot together $\mathbb{V}_{\pm 100}, \mathbb{V}_{\pm 200}, V_{1/10}, V_{100}$ ($\mathbb{V}_{100}$ and $\mathbb{V}_{200}$ are almost same, as well as $\mathbb{V}_{-100}$ and $\mathbb{V}_{-200}$), shown in Fig. 3.

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