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Partial Differential Equations

On Poincaré's and J.L. Lions' lemmas

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Abstract

Let Ω be a bounded, connected and simply connected open subset of $\mathbb{R}^N$ with a Lipschitz continuous boundary. It is shown that an irrotational vector field whose components are in $H^{-1}(\Omega)$ is the gradient of a function in $L^2(\Omega)$. It is also shown that this generalization of a classical lemma of Poincaré is equivalent to a well-known lemma of J.L. Lions. ***To cite this article: S. Kesavan, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur les lemmes de Poincaré et de J.L. Lions. Soit Ω un ouvert borné de $\mathbb{R}^N$ connexe et simplement connexe à frontière lipschitzienne. On montre qu'un champ vectoriel à composantes dans $H^{-1}(\Omega)$ dont le rotationnel est nul est le gradient d'une fonction dans $L^2(\Omega)$. On montre que cette généralisation d'un lemme classique de Poincaré est equivalent à un lemme très connu de J.L. Lions. ***Pour citer cet article : S. Kesavan, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Soit Ω un ouvert borné de $\mathbb{R}^N$, $N \geqslant 2$, connexe et simplement connexe à frontière lipschitzienne au sens de Nečas [5]. Un lemme classique de Poincaré affirme qu'un champ vectoriel à composantes dans $C^1(\Omega)$ dont le rotationnel est nul est le gradient d'une fonction dans $C^2(\Omega)$. Ce résultat à été généralisé par Girault et Raviart [4] au cas des champs dans $\mathbf{L}^2(\Omega)$ et, plus récemment, par Ciarlet et Ciarlet Jr. [2] au cas des champs dans $\mathbf{H}^{-1}(\Omega)$ lorsque $N = 3$. Nous allons démontrer ce dernier résultat pour $\Omega \subset \mathbb{R}^N$, pour tout entier $N \geqslant 2$.

Théorème 0.1. *Soit* $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, *un ouvert borné connexe et simplement connexe à frontière lipschitzienne. Soit* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$. *Alors, les énoncés suivants sont equivalents* :

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(i) $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$.
(ii) $\mathbf{h} = \mathbf{grad}\, p$ *où* $p \in L^2(\Omega)$.
(iii) $\langle \mathbf{h}, \mathbf{w}\rangle = 0$ *pour tout* $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ *tel que* $\operatorname{div} \mathbf{w} = 0$, *où* $\langle \cdot, \cdot \rangle$ *désigne le produit de dualité entre* $\mathbf{H}^{-1}(\Omega)$ *et* $\mathbf{H}_0^1(\Omega)$.

Idée de la démonstration. On reprend en partie la démarche de Ciarlet et Ciarlet Jr. [2]. On sait (cf. Girault et Raviart [4]) qu'ils existe $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ et $p \in L^2(\Omega)$ tels que

$$-\Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{h} \quad \text{dans } \mathbf{H}^{-1}(\Omega) \quad \text{et} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{dans } L^2(\Omega). \tag{1}$$

On utilise les hypothèses et l'hypoellipticité du laplacien pour en déduire facilement que $\Delta\mathbf{u} \in \mathbf{C}^\infty(\Omega)$ et que $\mathbf{curl}\,\Delta\mathbf{u} = \mathbf{0}$. Donc, d'après le lemme classique de Poincaré, on en déduit que $\Delta\mathbf{u} = \mathbf{grad}\,\tilde{p}$ où $\tilde{p} \in C^\infty(\Omega)$. Une application du lemme de Lions (cf. Duvaut et Lions [3] ou Amrouche et Girault [1]) nous permet de déduire que $\tilde{p} \in L^2(\Omega)$ et on conclut ainsi la démonstration de (i) implique (ii). La reciproque, ainsi que la démonstration de (ii) implique (iii) sont immédiates. Pour la démonstration de (iii) implique (ii), voir Girault et Raviart [4]. □

Remarque 1. L'equivalence (ii) ⇔ (iii) est connue sous le nom de *théorème de de Rham*. On n'a pas besoin de la simple connexité de Ω pour cette equivalence.

Corollaire 0.2. *Soit* $\Omega \subset \mathbb{R}^N$ *comme ci-dessus. Alors les énoncés suivants sont equivalents* :

(i) *Lemme de Poincaré dans* $\mathbf{H}^{-1}(\Omega)$: *Si* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ *vérifie* $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$, *alors* $\mathbf{h} = \mathbf{grad}\, p$, *où* $p \in L^2(\Omega)$.
(ii) *Lemme de Lions* : *Si* $f \in \mathcal{D}'(\Omega)$ *est tel que* $\mathbf{grad}\, f \in \mathbf{H}^{-1}(\Omega)$, *alors* $f \in L^2(\Omega)$.

Démonstration. D'une part, on a vu dans la démonstration précédente que (ii) implique (i). D'autre part, puisque $\mathbf{curl}\,\mathbf{grad}\, f = \mathbf{0}$, on déduit du théorème précédent que $\mathbf{grad}\, f = \mathbf{grad}\, p$, où $p \in L^2(\Omega)$ et donc $f = p$ à une constante additive près. □

1. Introduction

Let Ω be a bounded, connected and simply connected open subset of $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz continuous boundary in the sense of Nečas [5]. A classical *lemma of Poincaré* states that an irrotational vector field with components in $C^1(\Omega)$ is the gradient of a function in $C^2(\Omega)$.

When $\Omega \subset \mathbb{R}^3$, this has been generalised to irrotational vector fields with components in $L^2(\Omega)$ (as gradients of $H^1(\Omega)$ functions) by Girault and Raviart [4], and, more recently, to irrotational vector fields with components in $H^{-1}(\Omega)$ (as gradients of $L^2(\Omega)$ functions) by Ciarlet and Ciarlet Jr. [2].

Not only are such investigations interesting *per se*, but these results are also important in the study of solutions to the Stokes problem in fluid mechanics and in characterizing those symmetric second order tensors with components in $L^2(\Omega)$ that can arise as linearized strain tensors of displacement fields in elasticity (cf. [2]).

The aim of this Note is to give a proof of the result of [2] in any space dimension and to show that this result is equivalent to a well-known *lemma of Lions*. In all that follows, vectors in $\mathbb{R}^N$ and vector spaces whose elements are vector fields in $\mathbb{R}^N$ are denoted by boldface letters.

2. The main result

Theorem 2.1. *Let* Ω *be a bounded, connected and simply connected open subset of* $\mathbb{R}^N$, $N \geqslant 2$, *with a Lipschitz continuous boundary. Let* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$. *Then, the following are equivalent*:

(i) $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$.
(ii) $\mathbf{h} = \mathbf{grad}\, p$, *where* $p \in L^2(\Omega)$.
(iii) $\langle \mathbf{h}, \mathbf{w}\rangle = 0$, *for all* $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ *with* $\operatorname{div} \mathbf{w} = 0$, *where* $\langle \cdot, \cdot \rangle$ *denotes the duality pairing between* $\mathbf{H}^{-1}(\Omega)$ *and* $\mathbf{H}_0^1(\Omega)$.

Proof. First, we establish the implication (i) ⇒ (ii). To begin with, we follow the same argument as in Ciarlet and Ciarlet Jr. [2]. We know that there exist $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $p \in L^2(\Omega)$ such that (1) holds (cf. Girault and Raviart [4]). Taking the '**curl**' of the first relation, we get

$$\mathbf{curl}(\Delta \mathbf{u}) = \Delta(\mathbf{curl}\,\mathbf{u}) = \mathbf{0}.$$

The *hypoellipticity of the Laplace operator* then implies that $\mathbf{curl}\,\mathbf{u} \in \mathbf{C}^\infty(\Omega)$. Thus, using the summation convention on repeated indices, we see that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \in C^\infty(\Omega),$$

which can be rewritten as

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) \in C^\infty(\Omega).$$

However, the second term above vanishes since $\operatorname{div} \mathbf{u} = 0$. Thus $\Delta \mathbf{u} \in \mathbf{C}^\infty(\Omega)$ (this, in turn, implies that $\mathbf{u} \in \mathbf{C}^\infty(\Omega)$, but we will not need to use this information).

Thus, $\Delta \mathbf{u}$ is a smooth and irrotational vector field, and, by the *classical Poincaré lemma*, it is the gradient of a smooth scalar field $\tilde{p}$. Thus we see that $\mathbf{h} = \mathbf{grad}(p - \tilde{p})$.

Now, $\tilde{p}$ is a distribution whose gradient is in $\mathbf{H}^{-1}(\Omega)$ and so, by the *lemma of Lions* (cf. Duvaut and Lions [3] for a proof in case of smooth boundaries and Amrouche and Girault [1] for Lipschitz continuous boundaries), we deduce that $\tilde{p} \in L^2(\Omega)$ as well and this completes this part of the proof.

The implications (ii) ⇒ (i) and (ii) ⇒ (iii) are clear. The implication (iii) ⇒ (ii) has been proved by Girault and Raviart [4]. □

Remark 1. The implication (iii) ⇒ (ii), proved by Girault and Raviart [4], is a consequence of the fact that $\mathbf{grad} : L^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is an operator with closed range and is an important step in establishing the existence of a solution to (1). The equivalence (ii) ⇔ (iii) is well known in the literature and is often referred to as *de Rham's theorem*. It is also important to observe that this equivalence holds even when Ω is not simply connected.

Remark 2. We can easily recover the '$\mathbf{L}^2$ version' of the Poincaré lemma of Girault and Raviart [4] in all space dimensions. Indeed, since we already know by the above result that $\mathbf{h} = \mathbf{grad}\, p$ for some $p \in L^2(\Omega)$, we immediately conclude that $p \in H^1(\Omega)$ if $\mathbf{h} \in \mathbf{L}^2(\Omega)$.

We conclude with the following result.

Corollary 2.2. *Let* Ω *be a bounded, connected and simply connected open subset of* $\mathbb{R}^N$ *with Lipschitz continuous boundary. The following statements are equivalent*:

(i) *Poincaré lemma in* $\mathbf{H}^{-1}$: *If* $\mathbf{h} \in \mathbf{H}^{-1}(\Omega)$ *is such that* $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$, *then* $\mathbf{h} = \mathbf{grad}\, p$ *for some* $p \in L^2(\Omega)$.
(ii) *Lemma of Lions in* $\mathbf{H}^{-1}$: *If* $f \in \mathcal{D}'(\Omega)$ *is a distribution such that* $\mathbf{grad}\, f \in \mathbf{H}^{-1}(\Omega)$, *then* $f \in L^2(\Omega)$.

Proof. We saw in the proof of Theorem 2.1 that the lemma of Lions was used to infer that $\tilde{p}$ belonged to $L^2(\Omega)$. Thus, in effect, Theorem 1 shows that (ii) implies (i).

Conversely, if **grad** $f \in \mathbf{H}^{-1}(\Omega)$, then its **curl** vanishes, so that we have

$$\mathbf{grad}\, f = \mathbf{grad}\, p$$

for some $p \in L^2(\Omega)$, by Theorem 2.1. Thus, $f = p$ up to a constant function and the proof is complete. □

Korn's inequality, which is a fundamental inequality in the study of the well-posedness of the equations of linearized elasticity, is usually proved via a crucial application of Lions' lemma. Ciarlet and Ciarlet Jr. [2] discovered a new proof of this inequality using the Poincaré lemma. As a consequence, they were able to characterize those symmetric matrix fields in $\mathbf{L}^2(\Omega)$ which are linearized strain tensors of displacement fields. In view of the corollary above, this seems now quite natural.

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