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Partial Differential Equations

Partial regularity for homogeneous complex Monge–Ampere equations

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Abstract

In this Note, we establish a new partial regularity theory on certain homogeneous complex Monge–Ampere equations. This partial regularity theory is obtained by studying foliations by holomorphic disks and their relation to these equations. *To cite this article: X.X. Chen, G. Tian, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

Régularité partielle pour des équations de Monge–Ampère complexes. Dans cette Note, on établit un nouveau résultat de régularité partielle pour certaines équations complexes de Monge–Ampère. On obtient ces résultats en étudiant des feuilletages par des disques holomorphes et leurs relations avec ces équations. *Pour citer cet article : X.X. Chen, G. Tian, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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At the end of [3], we introduced the notion of almost smooth solutions. In this Note, we briefly discuss how to prove the existence of almost smooth solutions of the following Dirichlet problem for the homogeneous complex Monge–Ampere equation:

$$(\pi_2^* \omega + \partial\bar{\partial}\phi)^{n+1} = 0 \quad \text{on } \Sigma \times M, \quad \phi|_{\partial\Sigma \times M} = \psi, \quad (1)$$

where Σ is the unit disc in \mathbb{C} .

Theorem 1. *For a generic boundary map $\psi : \partial\Sigma \rightarrow \mathcal{H}_\omega$, there exists a unique almost smooth solution ϕ of (1).*

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The partial regularity in Theorem 1 is sharp since we do have examples of solutions with singularity for (1). This theorem is proved by establishing a foliation by holomorphic discs with mild singularity.

First we recall a construction from [7]: associate a complex symplectic manifold $\mathcal{W}_{[\omega]}$ to each Kähler class $[\omega]$: Let $\{U_i\}$ be a covering of M such that $\omega|_{U_i} = \sqrt{-1}\partial\bar{\partial}\rho_i$, we identify $(x, v_i) \in T^*U_i$ with $(y, v_j) \in T^*U_j$ if $x = y \in U_i \cap U_j$ and $v_i = v_j + \partial(\rho_i - \rho_j)$, then $\mathcal{W}_{[\omega]}$ consists of all these equivalence classes of $[x, v_i]$. The complex structure on T^*M induces an natural complex structure on $\mathcal{W}_{[\omega]}$ and there is also a canonical holomorphic 2-form Ω on $\mathcal{W}_{[\omega]}$, in terms of local coordinates z_α, ξ_α ($\alpha = 1, \dots, n$) of T^*U_i ,

$$\Omega = dz_\alpha \wedge d\xi_\alpha.$$

Now for any $\varphi \in \mathcal{H}_{[\omega]}$, we can associate a complex submanifold Λ_φ in $\mathcal{W}_{[\omega]}$: for any open subset U on which ω can be written as $\sqrt{-1}\partial\bar{\partial}\rho$, we define $\Lambda_\varphi|_U$ = the graph of $\partial(\rho + \varphi)$. Clearly, this Λ_φ is independent of the choice of U . A straightforward computation shows

$$\Omega|_{\Lambda_\varphi} = -\sqrt{-1}\omega_\varphi, \tag{2}$$

that is, $\text{Re}(\Omega)|_{\Lambda_\varphi} = 0$ and $-\text{Im}(\Omega)|_{\Lambda_\varphi} = \omega_\varphi > 0$. This means that Λ_φ is an exact Lagrangian symplectic submanifold of $\mathcal{W}_{[\omega]}$, i.e., it is Lagrange w.r.t. $\text{Re}(\Omega)$ while it is symplectic w.r.t. $\text{Im}(\Omega)$. Conversely, given an exact Lagrangian symplectic submanifold Λ of $\mathcal{W}_{[\omega]}$, we can construct a smooth function φ such that $\Lambda = \Lambda_\varphi$. Hence, Kähler metrics with Kähler class $[\omega]$ are in one-to-one correspondence with exact Lagrangian symplectic submanifolds in $\mathcal{W}_{[\omega]}$.

Let ψ be a smooth function on $\partial\Sigma \times M$ such that $\psi(\tau, \cdot) \in \mathcal{H}_{[\omega]}$ for any $\tau \in \partial\Sigma$. Define

$$\bar{\Lambda}_\psi = \{(\tau, v) \in \partial\Sigma \times \mathcal{W}_{[\omega]} \mid v \in \Lambda_{\psi(\tau, \cdot)}\}. \tag{3}$$

One can show that $\bar{\Lambda}_\psi$ is a totally real submanifold in $\Sigma \times \mathcal{W}_{[\omega]}$.

It is proved in [7] (also see [5]) states that there is a one-to-one correspondence between smooth solutions ϕ of (1) and holomorphic foliations of $\Sigma \times M$ induced by holomorphic discs $h_x: \Sigma \mapsto \mathcal{W}_{[\omega]}$ ($x \in M$) with boundary in $\bar{\Lambda}_\psi$ such that all leaves are transversal to M and $\pi(h_x(0)) = x$.

One of our crucial observations is that Semmes’ arguments can be made local along super-regular holomorphic discs. We will introduce the notion of nearly smooth foliations and show that they correspond to almost smooth solutions. Then Theorem 1 will be proved by constructing a nearly smooth foliation for a generic boundary value.

Given a boundary value ψ , we denote by \mathcal{M}_ψ the corresponding moduli space of holomorphic discs. First it follows from the Index theorem that the expected dimension of this moduli is $2n$. Recall that a holomorphic disc u is called *regular* if the linearized $\bar{\partial}$ -operator $\bar{\partial}_u$ has vanishing cokernel. The moduli space is smooth near a regular holomorphic disc. Following [5], we call u *super-regular* if there is a basis s_1, \dots, s_{2n} of the kernel of $\bar{\partial}_u$ such that $d\pi(s_1)(x), \dots, d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma$, where $\pi: \mathcal{W}_{[\omega]} \mapsto M$ is the natural projection. We call u *almost super-regular* if $d\pi(s_1)(x), \dots, d\pi(s_{2n})(x)$ span $T_{u(x)}M$ for every $x \in \Sigma \setminus \partial\Sigma$. Clearly, the set of super-regular discs is open.

Definition 2. Suppose $\psi: \partial\Sigma \rightarrow \mathcal{H}_\omega$ is given. A nearly smooth foliation of \mathcal{M}_ψ associated to the boundary map ψ is given by an open subset $\mathcal{U}_\psi \subset \mathcal{M}_\psi$ of super-regular discs whose images in $\Sigma \times M$ give rise to a foliation on an open-dense subset $\mathcal{V}_\psi \subset \Sigma \times M$ satisfying:

- (i) This foliation can be extended to be a continuous foliation of $\tilde{\mathcal{V}}_\psi \subset \Sigma_0 \times M$ by holomorphic discs whose complement is locally extendable;²
- (ii) The extended foliation admits a continuous lifting in $\Sigma \times \mathcal{W}_M$;
- (iii) The foliation in \mathcal{V}_ψ is uniformly transversal to any vertical fiber $\{z\} \times M$.

² A closed subset $S \subset \Sigma \times M$ is *locally extendable* if for any continuous function in $\Sigma \times M$ which is $C^{1,1}$ on $\Sigma \times M \setminus S$ can be extended to a $C^{1,1}$ function on $\Sigma \times M$. Note that any subset of codimension 2 or higher is automatically locally extendable.

Theorem 3. *For a generic boundary value ψ , an almost smooth solution of (1) corresponds to an nearly smooth foliation.*

Because of this theorem, Theorem 1 follows from the following:

Theorem 4. *For a generic boundary value ψ , there is a nearly smooth foliation generated by an open set of the corresponding moduli space \mathcal{M}_ψ .*

Now we outline the proof of Theorem 4. Let ψ be a generic boundary value such that \mathcal{M}_ψ is smooth. This follows from a result of Oh [6] on transversality. By a similar (but different) transversality argument, one can show that there is a generic path ψ_t ($0 \leq t \leq 1$) such that $\psi_0 = 0$ and $\psi_1 = \psi$ and the total moduli space $\widetilde{\mathcal{M}} = \bigcup_{t \in [0,1]} \mathcal{M}_{\psi_t}$ is smooth, moreover, we may assume that \mathcal{M}_{ψ_t} are smooth for all t except finitely many t_1, \dots, t_N where the moduli space may have isolated singularities. First we observe that \mathcal{M}_0 has at least one component which gives a foliation for $\Sigma \times M$. We want to show that this component will deform to a component of \mathcal{M}_ψ which generates a nearly smooth foliation. We will use the continuity method. Assume that ϕ is the unique $C^{1,1}$ -solution of (1) with boundary value ψ_t for some $t \in [0, 1]$. Let f be any holomorphic disc in the component of \mathcal{M}_{ψ_t} which generates the corresponding foliation.

It follows from an extension of Gromov’s compactness theorem, any sequence of holomorphic discs with uniformly bounded area has a subsequence which converges to a holomorphic disc together with finitely many bubbles. This is still true in our case, even though we can only show a uniform area bound on the image of any holomorphic disc $f \in \mathcal{M}_{\psi_t}$ in $\Sigma \times M$ instead of f itself. These bubbles which occur in the interior are holomorphic spheres, while bubbles in the boundary might be holomorphic spheres or discs.

For a fixed totally real submanifold, holomorphic bubbles cannot occur in the boundary. If a sequence of totally real submanifolds converges to a given totally real submanifold, there are two limiting processes, one concerns how fast the bubbles form and move to the boundaries of discs, while the other is how fast the sequence of totally real submanifolds approaches to the limiting submanifold. The uniform $C^{1,1}$ bound on ϕ can be used to show that the two limiting processes are exchangeable. Consequently, there are no bubbles along the boundary.

A solution of (1) can be regarded as a twisted harmonic map from Σ into the infinite dimensional space \mathcal{H}_ω (cf. [5]). According to [1], this infinite dimensional space \mathcal{H}_ω is non-positively curved in the sense of Alexandrov. Heuristically speaking, we can rule out the possibility of interior bubbles by exploring this curvature condition. Indeed, there is a rigorous proof for this fact. We refer the readers to [4] for a detailed proof.

Since there are no bubbles either in the boundary or interior of Σ , the Fredholm index of holomorphic discs is invariant in limiting process. This is an important fact needed in our using deformation theory.

In order to get a nearly smooth foliation, we need to prove that the moduli space has an open set of super-regular holomorphic discs for each t . First we observe that the set of super-regular discs is open. Moreover, using the transversality arguments, one can show that for a generic path ψ_t , the closure of all super-regular discs in each \mathcal{M}_{ψ_t} is either empty or forms an irreducible component. It implies the openness. It remains to prove that each moduli space has at least one super-regular disc. It is done by using capacity estimate which we explain briefly in the following.

Consider the bundle $\mathcal{E} = \pi_2^* TM$ over $\Sigma \times M$. Each almost smooth solution ϕ of (1) induces an Hermitian metric on $\mathcal{E}|_{\mathcal{V}_\psi}$, where \mathcal{V}_ψ was defined in Definition 2. If f is a super-regular disc, then \mathcal{E} pulls back to a Hermitian bundle over Σ with fiber $T_{f(z)}M$ and metric $\omega_{\phi(z,\cdot)}(f(z))$ over $z \in \Sigma$. It turns out that the curvature of this Hermitian bundle is non-positive. This fact plays a crucial role in our work. More precisely, we have

Lemma 5. *Let ϕ be a solution of (1) and f be a super-regular holomorphic disc as above, then the curvature form F of the metric g_ϕ described above is given by $g_\phi(F(u), v) = -g_\phi(u(\overline{\partial}_z f), v(\overline{\partial}_z f))$, $u, v \in TM$. In particular, the curvature is non-positive. Moreover, the foliation is holomorphic along f if and only if the curvature vanishes.*

The determinant $\wedge^n \mathcal{E}$ restricts to a Hermitian line bundle over any given super-regular disc. The corresponding Hermitian metric, denoted by $f^* \omega_\phi^n$, at z is $\omega_{\phi(z,\cdot)}^n(f(z))$. An immediate corollary of above lemma is that the

curvature of this line bundle is non-positive.³ Moreover, there are constants C_1, C_2 which depend only on the background metric ω such that

$$\Delta \left(\log \frac{f^* \omega_\phi^n}{f^* \omega^n} + C_1 \varphi \right) \geq 0, \quad \text{and} \quad \Delta \left(\log \frac{f^* \omega_\phi^n}{f^* \omega^n} + C_2 \varphi \right) \leq -\text{tr}(F), \quad (4)$$

where Δ denotes the standard Laplacian operator on Σ and $\varphi(z) = \phi(z, f(z))$. It follows that $\log \frac{f^* \omega_\phi^n}{f^* \omega^n} + C_1 \varphi$ is subharmonic and uniformly bounded on the boundary $\partial \Sigma$. The $C^{1,1}$ -estimate in [2] implies that this function is uniformly bounded from above. Moreover, the difference of two functions $\log \frac{f^* \omega_\phi^n}{f^* \omega^n} + C_1 \varphi$ and $\log \frac{f^* \omega_\phi^n}{f^* \omega^n} + C_2 \varphi$ is uniformly bounded. In addition, we have

$$-\Delta \text{tr}(F) \geq \frac{2}{n} (-\text{tr}(F))^2. \quad (5)$$

According to a result of Osseman, we can derive an interior estimate on $\text{tr}(F)$ (see [4] for details). Applying this estimate on $\text{tr}(F)$ to the above equations, we can derive a Harnack-type inequality $\frac{f^* \omega_\phi^n}{f^* \omega^n}$ in the interior of Σ .

Now let us introduce the notion of Capacity for super-regular holomorphic discs:

Definition 6. For any super-regular disc f in an moduli space \mathcal{M}_ψ , we define its capacity by $\text{Cap}(f) = \int_\Sigma \frac{f^* \omega^n}{f^* \omega_\phi^n} \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$.

Using the Harnack-type inequality mentioned above, one can control the lower bound of $\frac{f^* \omega_\phi^n}{f^* \omega^n}$ in the interior of σ in terms of upper bound of the capacity of f . This has an important corollary for compactness of super-regular discs with uniformly bounded capacity.

Theorem 7. Let f_i be any sequence of super-regular discs in \mathcal{M}_{ψ_i} which converges smoothly to an embedded disc f_∞ in $\mathcal{M}_{\psi_{t_\infty}}$. If the capacities $\text{Cap}(f_i)$ are uniformly bounded, then the limiting disc f_∞ is also super-regular.

In fact, Lemma 5 was already needed when we extended Semmes' correspondence to almost smooth solutions of (1) and nearly smooth foliations. For this local extension, we first construct smooth solutions of (1) along super-regular leaves and then glue them together to a solution ϕ on an open and dense subset $\mathcal{V}_\phi \subset \Sigma \times M$, but we need to establish a global $C^{1,1}$ -bound of ϕ on \mathcal{V}_ϕ . Once this bound is established, the maximum principle implies that ϕ coincides with the solution in [2]. The $C^{1,1}$ -bound of ϕ follows from the following

Theorem 8. For any global holomorphic section $s : \Sigma \rightarrow \mathcal{E}$, the norm of s with respect to g_ϕ achieves its maximum value at the boundary of the disc.

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³ This fact using a different method, was known to the first name author and S.K. Donaldson in 1998 while both of them were visiting Stanford University.