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Partial Differential Equations

Efficiency of approximate boundary conditions for corner domains coated with thin layers

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Abstract

In this Note, we consider an interface problem posed in a bounded domain with thin layer. In the case of a smooth domain, approximate boundary conditions (also called impedance conditions) are known to approximate in a precise way the effect of the layer, as its thickness goes to zero. We investigate here the efficiency of such conditions when the domain has a corner; we show that it deteriorates when the opening of the corner angle grows, giving optimal estimates thanks to multiscale asymptotic expansions. Numerical results are given, which illustrate these estimates. ***To cite this article: G. Vial, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Performance des conditions aux limites approchées pour des domaines à coin recouverts de couches minces. On considère un problème de transmission posé dans un domaine borné recouvert d'une couche mince. Dans le cas d'un domaine régulier, il est connu qu'on peut remplacer efficacement l'effet de la couche par des conditions aux limites approchées (aussi appelées conditions d'impédance), quand son épaisseur tend vers zéro. On s'intéresse ici à l'efficacité de telles conditions quand le domaine présente un coin ; on montre qu'elle se dégrade lorsque l'ouverture du coin augmente. Grâce à une analyse multi-échelle, on obtient des estimations optimales, qui sont illustrées par des calculs numériques. ***Pour citer cet article : G. Vial, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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On considère un problème de transmission de type div$(a\nabla)$ dans un domaine bidimensionnel recouvert d'une couche mince d'épaisseur ε, voir Fig. 1 (le coefficient a vaut 1 dans le domaine intrieur Ω_{int} et α dans la couche $\Omega^{\varepsilon}_{\text{ext}}$). La difficulté à approcher correctement la solution de ce problème d'un point de vue numérique a conduit de nombreux auteurs à s'intéressser à l'identification de conditions aux limites approchée (cf. [5,1] par exemple). La justification de la convergence du problème approché correspondant vers le problème de transmission (quand l'épaisseur ε tend vers 0) n'a été faite que dans le cas où le domaine est régulier. Nous proposons ici d'examiner le cas d'un domaine présentant un coin d'ouverture angulaire ω, en utilisant les outils de l'analyse asymptotique multi-échelle. Précisément, on construit un développement asymptotique en puissances de ε pour la solution de chacun des deux problèmes et on en compare les premiers termes.

Dans le cas où le domaine est régulier, la solution du problème de transmission admet dans le domaine intrieur Ω_{int} le développement $u_{\varepsilon,\text{int}} \sim \sum \varepsilon^n u^n_{\text{int}}$, avec u^n_{int} indépendant de ε. Quant à celui de la solution du problème avec condition aux limites approchées, il s'écrit $v_\varepsilon \sim \sum \varepsilon^n v^n$. Comme les premiers termes coïncident pour $n \leqslant 2$, on obtient une estimation optimale en ε^3 de la différence en norme d'énergie entre $u_{\varepsilon,\text{int}}$ et v_ε.

Pour un domaine à coin, les développements asymptotiques doivent être corrigés pour tenir compte des singularités de coin qui apparaissent à chaque étape de la construction : des puissances non-entières de ε apparaissent, ainsi que de termes correctifs localisés au coin. Ces derniers consistent en des *profils* construits dans le domaine infini Q (cf. Fig. 1), ils opèrent en la variable rapide $\frac{x}{\varepsilon}$. La construction de ces développements utilise fortement la décomposition des termes en parties régulière et singulière (voir [6]), la première pouvant être traitée comme dans le cas d'un domaine régulier (voir [2] ou [4] pour des techniques analogues, et [8] ou [3] pour les détails de cette construction). La comparaison des premiers termes permet d'obtenir l'estimation optimale

$$\|u_{\varepsilon,\text{int}} - v_\varepsilon\|_{\mathrm{H}^1(\Omega_{\text{int}})} \sim \varepsilon^{\min(3,2\pi/\omega)}.$$

Ainsi il apparaît que si la condition aux limites approchée est aussi performante que dans le cas régulier pour de faibles ouvertures ω, son efficacité se dégrade quand l'angle ω augmente.

Enfin, des calculs numériques ont été réalisés à l'aide de la bibliothèque éléments finis MÉLINA (voir [7]), qui illustrent l'optimalité des résultats théoriques énoncés, voir Fig. 3. Notons qu'une grande précision est requise, et que l'usage d'une méthode adaptée à la fois la présence de singularités et à la couche limite dans le domaine extérieur est nécessaire. La solution adoptée consiste à utiliser une méthode en p-version, couplée à un raffinement du maillage au voisinage du coin, et des éléments anisotropes dans la couche (cf. Fig. 2).

1. Introduction

In the following, $\mathrm{H}^s(\Omega)$ denotes the standard Sobolev space, endowed with its natural norm $\|\cdot\|_{s,\Omega}$.

Let $\Omega_{\text{int}} \subset \mathbb{R}^2$ be a bounded domain with boundary Γ, infinitely smooth except at the origin O. In the neighborhood of this point O, we assume that Ω_{int} coincides with a plane sector of opening ω ($\omega \neq 0, 2\pi$). For any $t \in \Gamma \neq 0$ let $\mathbf{n}(t)$ denote the unit outward normal at t. For $\varepsilon > 0$ small enough, let $\Omega^{\varepsilon}_{\text{ext}}$ be the layer of uniform thickness ε around Ω_{int} given by

$$\Omega^{\varepsilon}_{\text{ext}} = \bigl\{x \in \mathbb{R}^2;\ x = t + s\mathbf{n}(t), t \in \Gamma, s \in (0,\varepsilon)\bigr\}, \tag{1}$$

outside a neighborhood of O; near the origin O, $\Omega^{\varepsilon}_{\text{ext}}$ is also taken as a sector of opening ω at a distance ε from Γ, with vertex O^ε (see Fig. 1). We denote by Ω^ε the whole domain with layer: $\Omega^\varepsilon = \Omega_{\text{int}} \cup \Gamma \cup \Omega^{\varepsilon}_{\text{ext}}$.

In the following, we will refer to the situation $\omega \neq \pi$ as the 'corner case', whereas the 'smooth case' corresponds to $\omega = \pi$. We emphasize that all the results stated for the latter also apply if Γ is a smooth curve near O instead of a straight line.

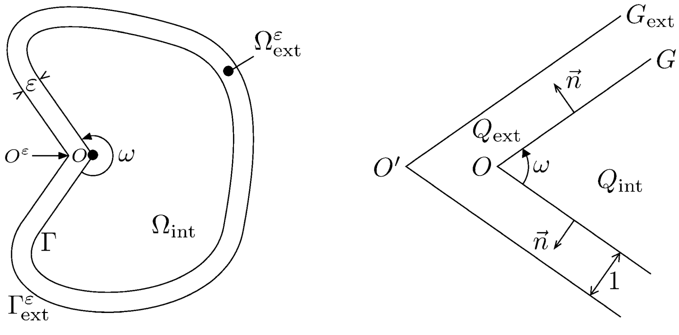

Fig. 1. The bounded domain Ω^ε (left) and the infinite domain Q (right), concave and convex cases. Le domaine borné Ω^ε (à gauche) et le domaine infini Q (à droite), cas concave et convexe.

We consider in the domain Ω^ε the following interface problem: Find u_ε, defined by $u_{\varepsilon,\rm int}$ in $\Omega_{\rm int}$ and $u_{\varepsilon,\rm ext}$ in $\Omega^\varepsilon_{\rm ext}$ satisfying the equations

$$\begin{cases} \alpha\Delta u_{\varepsilon,\rm int} = f_{\rm int} & \text{in } \Omega_{\rm int},\\ \Delta u_{\varepsilon,\rm ext} = 0 & \text{in } \Omega^\varepsilon_{\rm ext},\\ u_{\varepsilon,\rm int} - u_{\varepsilon,\rm ext} = 0 & \text{on } \Gamma,\\ \alpha\partial_n u_{\varepsilon,\rm int} - \partial_n u_{\varepsilon,\rm ext} = 0 & \text{on } \Gamma,\\ u_{\varepsilon,\rm ext} = 0 & \text{on } \Gamma^\varepsilon_{\rm ext}, \end{cases} \tag{2}$$

where ∂_n denotes the normal derivative (outer for $\Omega_{\rm int}$, inner for $\Omega^\varepsilon_{\rm ext}$). The right-hand side $f_{\rm int}$ does not depend on the parameter ε and is assumed sufficiently smooth. The real number α is positive: Eqs. (2) define a well-posed elliptic problem in $\mathrm{H}^1(\Omega^\varepsilon_{\rm ext})$ (the special case $\alpha = 1$ is not a genuine interface problem, but rather a domain perturbation).

From a numerical point of view, this interface problem requires the discretisation of the layer, which becomes a difficulty for very small thickness ε. For this reason, many mathematical works have been carried out, revealing approximate boundary problems only posed in the interior domain $\Omega_{\rm int}$, but involving ε-dependent boundary conditions on Γ (see e.g. [5,1]). For problem (2), the second-order approximate boundary problem reads

$$\begin{cases} \alpha\Delta v_\varepsilon = f_{\rm int} & \text{in } \Omega_{\rm int},\\ (1+\varepsilon^2\frac{c(x)}{2})v_\varepsilon + \alpha\varepsilon\partial_n v_\varepsilon = 0 & \text{on } \Gamma, \end{cases} \tag{3}$$

where $c(x)$ stands for the curvature of Γ at the point x. In the previous works, estimates of the convergence of v_ε towards $u_{\varepsilon,\rm int}$ are given (for exterior problems), under smoothness assumptions on the boundary Γ. The authors make use of a variational method, involving low-order polynomial test functions of the transversal coordinate in the layer. They prove the estimate $\|u_{\varepsilon,\rm int} - v_\varepsilon\|_{1,\Omega_{\rm int}} \leqslant C\varepsilon^{5/2}$.

This result can be improved using the tools of multi-scale asymptotic expansion:

Theorem 1.1. *In the 'smooth case', the solutions $u_{\varepsilon,\rm int}$ and v_ε, defined in problems* (2) *and* (3), *admit the following asymptotic expansions*:

$$u_{\varepsilon,\rm int} = \sum_{n=0}^{N} \varepsilon^n u^n_{\rm int} + r^N_{\varepsilon,\rm int} \quad \textit{and} \quad v_\varepsilon = \sum_{n=0}^{N} \varepsilon^n v^n + \rho^N_\varepsilon, \tag{4}$$

with the (*generically optimal*) *remainder estimates, with a constant $C = C(N, \Omega_{\rm int})$*:

$$\|r^N_{\varepsilon,\rm int}\|_{1,\Omega_{\rm int}} \leqslant C\varepsilon^{N+1}\|f_{\rm int}\|_{N,\Omega_{\rm int}} \quad \textit{and} \quad \|\rho^N_\varepsilon\|_{1,\Omega_{\rm int}} \leqslant C\varepsilon^{N+1}\|f_{\rm int}\|_{N,\Omega_{\rm int}}. \tag{5}$$

The detailed construction of the asymptotic expansion for $u_{\varepsilon,\,\mathrm{int}}$ is based on a dilation of the layer in the normal direction, it can be found in [8] or [3]. For v_ε, the term v^n is simply defined by $\Delta v^n = 0$ in Ω_{int} with Dirichlet boundary condition on Γ: $v^n = -\alpha\partial_n v^{n-1} - \frac{c(x)}{2} v^{n-2}$; the process being initialized with $v^0 \in \mathrm{H}^1_0(\Omega_{\mathrm{int}})$ satisfying $\alpha\Delta v^0 = f_{\mathrm{int}}$ and v^1 harmonic in Ω_{int} such that $v^1 = -\alpha\partial_n v^0$ on Γ.

A precise look at the first terms shows that $u^n_{\mathrm{int}} = v^n$ for $n = 0, 1, 2$. A direct consequence is

Corollary 1.2. *In the 'smooth case', the solutions $u_{\varepsilon,\,\mathrm{int}}$ and v_ε, defined in problems* (2) *and* (3)*, satisfy the following* (*generically optimal*) *estimate, with a constant $C = C(\Omega_{\mathrm{int}})$*:

$$\|u_{\varepsilon,\,\mathrm{int}} - v_\varepsilon\|_{1,\Omega_{\mathrm{int}}} \leqslant C\varepsilon^3 \|f_{\mathrm{int}}\|_{2,\Omega_{\mathrm{int}}}. \tag{6}$$

The main goal of this note is to investigate the 'corner case': we show that the approximation (6) no longer holds. A limitation due to the opening of the corner angle appears.

Theorem 1.3. *In the 'corner case', the solutions $u_{\varepsilon,\,\mathrm{int}}$ and v_ε, defined in problems* (2) *and* (3)*, satisfy the following* (*generically optimal*) *estimate, with a constant $C = C(\Omega_{\mathrm{int}})$*:

$$\|u_{\varepsilon,\,\mathrm{int}} - v_\varepsilon\|_{1,\Omega_{\mathrm{int}}} \leqslant C P(\log\varepsilon)\varepsilon^{\min(3,\pi/\omega)} \|f_{\mathrm{int}}\|_{2,\Omega_{\mathrm{int}}}, \tag{7}$$

where P is a polynomial of degree at most 3.

Remark 1. Problem (3) involves the curvature $c(x)$: in the 'corner case', it can still be defined as a discontinuous function on Γ almost everywhere. For polygonal domains, the approximate boundary condition simply reads $v_\varepsilon + \alpha\varepsilon\partial_n v_\varepsilon = 0$.

2. Tools for the proof of Theorem 1.3

The expansions (4) are no longer valid when Ω_{int} presents a corner point. They have to be corrected to take into account the contribution of the singularities around the corner point (see [2] or [4] for similar issues). Since the singular functions are localised near the corner, the correctors involve a cut-off function χ, equal to 1 in the neighbourhood of the origin and 0 away from this point. Non-integer exponents of ε also appear in the asymptotic expansion.

Theorem 2.1. *In the 'corner case', the solutions $u_{\varepsilon,\,\mathrm{int}}$ and v_ε, defined in problems* (2) *and* (3)*, admit the following asymptotic expansions*:

$$u_{\varepsilon,\,\mathrm{int}} = \sum_{\mu<N+1} \varepsilon^\mu u^\mu_{\mathrm{int}} + \chi(x) \sum_{\mu+\lambda<N+1} \varepsilon^{\mu+\lambda} c_\mu \mathfrak{K}^\lambda\left(\frac{x}{\varepsilon}\right) + r^N_{\varepsilon,\,\mathrm{int}}, \tag{8}$$

$$v_\varepsilon = \sum_{\mu<N+1} \varepsilon^\mu v^\mu_{\mathrm{int}} + \chi(x) \sum_{\mu+\lambda<N+1} \varepsilon^{\mu+\lambda} d_\mu \mathfrak{Z}^\lambda\left(\frac{x}{\varepsilon}\right) + \rho^N_\varepsilon, \tag{9}$$

with the remainder estimates (5)*. The sum over λ is extended to $\{\frac{q\pi}{\omega}, q \in \mathbb{N}^*\}$ and the indices μ cover the set $\mathbb{N} \cup \{\frac{h\pi}{\omega} + p;\ p \geqslant 0, h \geqslant 2\}$.*

The proof of this result makes use of the splitting of the successive terms into regular and singular part (see [6] for instance); the *profiles* $\mathfrak{K}^\lambda$ (respectively $\mathfrak{Z}^\lambda$) are constructed in the infinite sectorial domain Q (respectively Q_{int}), see Fig. 1. We give a few ideas for the construction of $\mathfrak{Z}^\lambda$ (for the details of the proof, see [8]): for $\lambda = \frac{q\pi}{\omega}$, the profile $\mathfrak{Z}^\lambda$ satisfies

$$\alpha\Delta\mathfrak{Z}^\lambda = 0 \quad \text{in } Q_{\mathrm{int}}, \quad \mathfrak{Z}^\lambda + \alpha\partial_n\mathfrak{Z}^\lambda = 0 \quad \text{on } G \quad \text{and} \quad \mathfrak{Z}^\lambda \sim \mathfrak{s}^\lambda \quad \text{as } |X| \to \infty, \tag{10}$$

where $\mathfrak{s}^\lambda$ is the singular function given in polar coordinates around the corner by $R^\lambda \sin(\lambda\theta)$; it solves $\Delta\mathfrak{s}^\lambda = 0$ in Q_{int} with homogeneous Dirichlet boundary conditions on G. The existence of such a profile $\mathfrak{Z}^\lambda$ follows from an algorithmic determination of the main terms at infinity, and a variational framework for problem (10) with non-zero date. The behaviour at infinity can be fully described using the Mellin transform, and compared with the counterpart $\mathfrak{K}^\lambda$ for the transmission problem. Precisely, we have for the difference of the profiles at infinity:

$$\mathfrak{K}^\lambda(X) - \mathfrak{Z}^\lambda(X) = \mathcal{O}\big(|X|^{\max(\lambda-3,-\pi/\omega)}\big) \quad \text{as } |X| \to +\infty, \tag{11}$$

and for the inner terms (arising from a cut-off error)

$$u^\mu_{\text{int}} = v^\mu \quad \text{for } \mu < \min\left(3, \frac{2\pi}{\omega}\right). \tag{12}$$

Altogether, we obtain the estimate in H^1-norm stated in Theorem 1.3.

3. Numerical results

In this section, we present numerical computations for the convergence of v_ε towards $u_{\varepsilon,\text{int}}$ as ε goes to zero, which illustrate the optimality of the result stated in Theorem 1.3. The domain is a polygon, with only two edges covered by a thin layer (this allows to concentrate around the common vertex, i.e. only one corner point); an example is given in Fig. 2. No singularity will appear around the other two vertices, thanks to a reflection principle (we impose Neumann boundary conditions on Γ_N). We choose the transmission coefficient α equal to 10, and $f_{\text{int}}(x,y) = 1$ if $x < -0.5$, 0 elsewhere (this data is not infinitely smooth, but the results still hold). According to Theorem 1.3, the expected convergence rate of the quantity $\|u_{\varepsilon,\text{int}} - v_\varepsilon\|_{1,\Omega_{\text{int}}}$ is: $\min(3, \frac{\pi}{\omega})$.

Since a high accuracy is needed, we used a mesh refinement near the corner point, combined with high order quadrangular elements ($\mathbb{Q}_8$). Moreover, only one layer of anisotropic elements is used in $\Omega^\varepsilon_{\text{ext}}$ (this is justified by the fact that the corner layer in the exterior domain is polynomial in the transversal variable).

On the left of Fig. 2, the error $\|u_{\varepsilon,\text{int}} - v_\varepsilon\|_{1,\Omega_{\text{int}}}$ is graphed against ε (log–log axes) for various values of the angle ω. After a pre-asymptotic behaviour, the curves become close to straight lines, whose slopes are estimated to produce the graph on the right. The latter shows a perfect correspondence between the theoretical convergence rate (solid line) and the estimated convergence rate (dots).

The computations have been carried out using the finite element library MÉLINA, see [7].

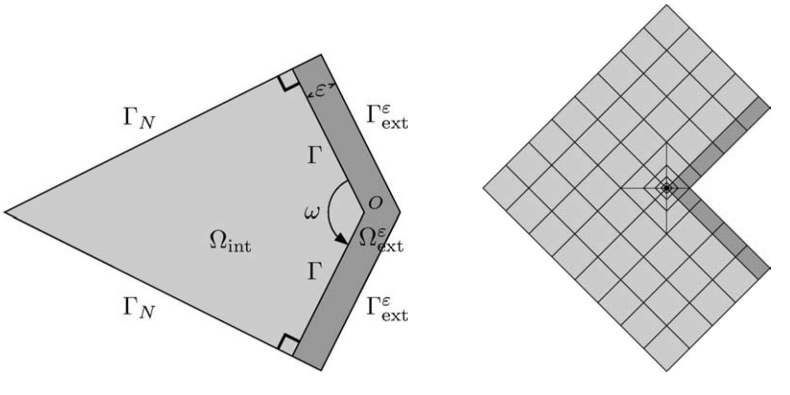

Fig. 2. Computational domain (example for $\omega < \pi$) and mesh (example for $\omega = \frac{3\pi}{2}$). Domaine de calcul (exemple pour $\omega < \pi$) et maillage (exemple pour $\omega = \frac{3\pi}{2}$).

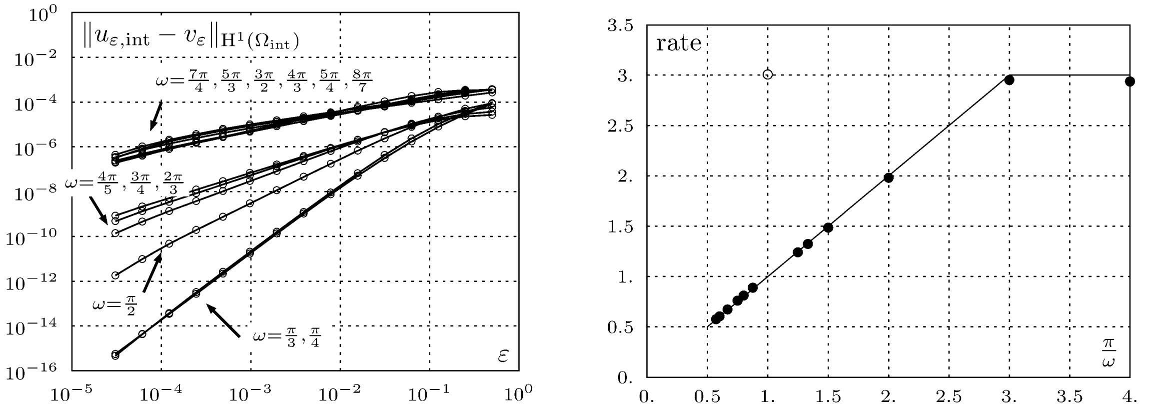

Fig. 3. On the left, each curve corresponds to a value of ω. On the right, the dots represent the estimated convergence rate of the quantity $\|u_{\varepsilon,\,\mathrm{int}} - v_\varepsilon\|_{1,\Omega_{\mathrm{int}}}$ and the solid line corresponds to the theoretical estimate of Theorem 1.3. À gauche; chaque courbe correspond à une valeur de ω. À droite, les points représentent le taux de convergence estimé de la quantité $\|u_{\varepsilon,\,\mathrm{int}} - v_\varepsilon\|_{1,\Omega_{\mathrm{int}}}$ et la ligne continue correspond à l'estimation théorique du Théorème 1.3.

4. Conclusion

The classical approximate boundary conditions loose of their efficiency in presence of corners in the domain. We have given a precise quantification of this efficiency, depending of the opening angle at the corner point. Our result shows the deterioration of the approximation for large angles, whereas we recover the same convergence rate for small angles than for a smooth domain. The use of multi-scale asymptotic expansions has allowed our obtaining sharp results and can also be applied in more complicated geometries, and for more elaborate boundary problems (in particular, Neumann external boundary conditions can be handled the same way).

References

[1] A. Bendali, K. Lemrabet, The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation, SIAM J. Appl. Math. 56 (6) (1996) 1664–1693.
[2] M. Bourlard, A. Maghnouji, S. Nicaise, L. Paquet, Asymptotic expansion of the solution of a mixed Dirichlet–Ventcel problem with a small parameter, Asymptotic Anal. 28 (3–4) (2001) 241–278.
[3] G. Caloz, M. Costabel, M. Dauge, G. Vial, Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer, Preprint 04-32, IRMAR.
[4] M. Costabel, M. Dauge, A singularly perturbed mixed boundary value problem, Commun. Partial Differential Equations 21 (1996) 1919–1949.
[5] B. Engquist, J.C. Nédélec, Effective boundary conditions for electromagnetic scattering in thin layers, Rapport interne 278, CMAP 1993.
[6] P. Grisvard, Boundary Value Problems in Non-Smooth Domains, Pitman, London, 1985.
[7] D. Martin. MÉLINA, bibliothèque de calculs éléments finis, http://perso.univ-rennes1.fr/daniel.martin/melina, 2003.
[8] G. Vial, Analyse multi-échelle et conditions aux limites approchées pour un problème de couche mince dans un domaine à coin, Thèse de doctorat 2840, Université de Rennes I, IRMAR, 2003.