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Algebraic Geometry/Topology

The fundamental group of an algebraic link ☆

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Abstract

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Résumé

Le groupe fondamental d'un entrelacs algébrique. On calcule le groupe fondamental d'un entrelacs algébrique. ***Pour citer cet article : O. Neto, P.C. Silva, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Soit Y un germe d'une courbe plane en un point o et Y_d, $1 \leqslant d \leqslant r$, ses composantes irréductibles. On choisit un système local de coordonnées (x, y) dans un polydisque ouvert X centré en o tel que le cône tangent de Y soit transversal à $\{x = 0\}$. Soit $y = \sum_\varepsilon a_{d,\varepsilon} x^\varepsilon$ le développement de Puiseux de Y_d, où $\varepsilon \in \mathbb{Q}$, $\varepsilon \geqslant 0$, et $a_{d,\varepsilon} \in \mathbb{C}$. On va associer à Y un arbre Ξ_Y dont l'ensemble des sommets est noté $\mathcal{V}_Y$. On utilise la terminologie généalogique. Étant donnés $1 \leqslant d, e \leqslant r$ et ε un nombre rationnel non négatif, on identifie (d, ε) avec (e, ε) si $a_{d,\nu} = a_{e,\nu}$ pour $\nu \in \mathbb{Q}$ et $0 \leqslant \nu \leqslant \varepsilon$. On dit que (une classe) (d, ε) est un *sommet de* Ξ_Y *avec un exposant* ε si $\varepsilon = 0$ ou si ε est un exposant caractéristique de Puiseux de Y_d ou s'il existe $e \neq d$ tel que $a_{d,\delta} = a_{e,\delta}$ pour $\delta < \varepsilon$ et $a_{d,\varepsilon} \neq a_{e,\varepsilon}$. On appelle $\phi = (d, 0)$ *la racine* de Ξ_Y. On dit que $w = (d, \delta)$ est un *fils* de $v = (d, \varepsilon)$ $(w > v)$ si $\delta > \varepsilon$ avec δ minimal. On dit que (d, ε) est une *tige* si $a_{d,\varepsilon} = 0$. On associe à chaque sommet $v = (d, \varepsilon) \in \mathcal{V}_Y$, non terminal, la branche

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$Y_v = \{y = \sum_{\nu<\delta} a_{d,\nu} x^\nu\}$ où $w = (d, \delta)$ est un fils quelconque de v. Si v est terminal on lui associe la branche $Y_v = Y_d$. Étant donné $0 < \eta \ll 1$ soit $K_v = Y_v \cap (\{x\colon |x| = \eta\} \times \mathbb{C})$ le nœud de Y_v.

Lemme 0.1. *On peut trouver $\eta > 0$ et des voisinages fermés N_v de K_v, $v \in \mathcal{V}_Y$, tels que $N_w \subset \operatorname{int}(N_v \setminus K_v)$ si $v < w$ et $N_v \cap N_w = \emptyset$ si $v \not< w$ et $w \not< v$.*

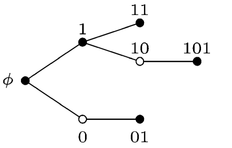
On dit que le système de voisinages (N_v) vérifiant les conditions ci-dessus est *un système torique pour Y*. On appelle *méridien standard* et *parallèle standard* de N_v une paire de courbes α_v et β_v sur ∂N_v telles que α_v, β_v soient homéomorphies à S^1, $\alpha_v \sim 0$, $\beta_v \sim K_v$ dans $H_1(N_v)$, $\ell(\alpha_v, K_v) = 1$ et $\ell(\beta_v, K_v) = 0$. Ici $\ell(\cdot,\cdot)$ désigne le nombre d'entrelacements dans une sphère d'homologie de dimension trois convenable. Si v est non terminal on dénote par $v_1, \ldots, v_{b_v}$ les fils de v qui ne sont pas des *tiges*, convenablement ordonnés. Ils existent des entiers positifs μ_v et ν_v, premiers entre eux, tels que $K_{v_i} \sim \mu_v \alpha_v + \nu_v \beta_v$ dans $H_1(N_v \setminus K_v)$. Les valeurs de μ_v et de ν_v correspondent, respectivement, à la multiplicité d'intersection et « l'indice de ramification » de Y_v et Y_{v_i}. Soient r_v, s_v des nombres entiers tels que $r_v \mu_v = s_v \nu_v + 1$.

Théorème 0.2. *Pour chaque germe de courbe plane Y, le groupe fondamental local de Y est presenté par les générateurs α_v, β_v, $v \in \mathcal{V}_Y$, et les relations $\beta_\phi = 1$, $[\alpha_v, \beta_v] = 1$ pour tous v,* (1)–(3) *et* (4).

Exemple 1. Soient $\rho = p/q > 3/2$ un nombre rationnel tel que $\operatorname{pgcd}(p, q) = 1$ et $r, s \in \mathbb{Z}$: $rp = sq + 1$. Soient Y une courbe plane et Y_{11}, Y_{101}, Y_{01} ses composantes irréductibles données, respectivement, par les développements de Puiseux $y = x^{3/2} + x^{7/4}$, $y = x^{3/2} + x^{5/2}$ et $y = x^\rho$. On représente les sommets de Ξ_Y qui correspondent à des *tiges* par des cercles blancs et les restant par des cercles noirs.

Avec les notations évidentes on a $\varepsilon_\phi = 0$, $\varepsilon_0 = \varepsilon_1 = 3/2$, $\varepsilon_{11} = \varepsilon_{10} = 7/4$, $\varepsilon_{101} = 5/2$, $\varepsilon_{01} = \rho$, $(\mu_\phi, \nu_\phi) = (3, 2)$, $(\mu_0, \nu_0) = (p, q)$, $(\mu_1, \nu_1) = (13, 2)$ et $(\mu_{1,0}, \nu_{1,0}) = (8, 1)$. Par le Théorème 0.2 le groupe fondamental local de Y est presenté par α_v, β_v, $v \in \{\phi, 1, 0, 11, 10, 01, 101\}$, vérifiant les relations $\beta_\phi = 1$, $[\alpha_v, \beta_v] = 1$ pour tous v, et les relations associées à chaque sommet non terminal de l'arbre Ξ_Y ci-dessous :

$$\begin{array}{rll}
\phi: & \alpha_\phi^3 \beta_\phi^2 = \alpha_1^6 \beta_1 = \alpha_0^3 \beta_0^2, & \alpha_1 \alpha_\phi \beta_\phi = \alpha_0 \beta_0, \\
0: & \alpha_0^p \beta_0^q = \alpha_{01}^{pq} \beta_{01}, & (\alpha_{01} \alpha_0^s \beta_0^r)^q = (\alpha_0^p \beta_0^q)^r, \\
1: & \alpha_1^{13} \beta_1^2 = \alpha_{11}^{26} \beta_{11} = \alpha_{10}^{13} \beta_{10}^2, & \alpha_{11} \alpha_1^6 \beta_1 = \alpha_{10}^6 \beta_{10}, \\
10: & \alpha_{10}^8 \beta_{10} = \alpha_{101}^8 \beta_{101}, & \alpha_{101} \alpha_{10}^7 \beta_{10} = \alpha_{10}^8 \beta_{10}.
\end{array}$$



1. Introduction and definitions

In this Note we generalize the results of Zariski, Kashiwara and Lê [8,3,4] on the computation of the local fundamental group of a plane curve. Ausina's algorithm [1] works in a more general framework but in order to generalize the results of [7] (see also [6]) we need a closed form presentation of the local fundamental group. Our computation of the fundamental group relies on a tree similar to Eggers's tree and on a decomposition that is related with the minimal Waldhausen decomposition of the 3-sphere adapted to the link of the curve (see [5]).

Throughout this Note we shall use Deligne's convention on the composition of paths, i.e., in $\alpha\beta$, we move along β in the first place. Let Ξ be a rooted tree with root ϕ. We say that a vertex w is a *child* of a vertex v ($w > v$) if w is connected to v by an edge and the path that connects w to ϕ contains v. We will use freely the usual genealogical terminology. We define an order in the set $\mathcal{V}$ of vertices of Ξ setting $o(\phi) = 0$ and $o(w) = o(v) + 1$ if $w > v$. We choose at most a child of each vertex v and call it a *shaft* of v. We call the number b_v of children of v that are not shafts, the *bifurcation index* at v.

Consider a germ Y at some point o of a plane curve with irreducible components Y_d, $1 \leqslant d \leqslant r$. We choose a system of local coordinates (x, y) on an open polydisc X centered at o such that the tangent cone of Y is transversal to $\{x=0\}$. Let $y=\sum_\varepsilon a_{d,\varepsilon}x^\varepsilon$ be a Puiseux expansion of Y_d, where $\varepsilon\in\mathbb{Q}$, $\varepsilon\geqslant 0$, and $a_{d,\varepsilon}\in\mathbb{C}$. We can assume that $a_{d,\delta}=a_{e,\delta}$ for all $\delta\leqslant\varepsilon$ if $\sum_{\delta\leqslant\varepsilon}a_{d,\delta}x^\delta$ and $\sum_{\delta\leqslant\varepsilon}a_{e,\delta}x^\delta$ parametrize the same curve. Let us associate to Y a tree $\varXi_Y$ with set of vertices $\mathcal{V}_Y$. If $1\leqslant d,e\leqslant r$ and ε is a nonnegative rational number, we identify (d,ε) and (e,ε) if $a_{d,\nu}=a_{e,\nu}$ for $\nu\in\mathbb{Q}$, $0\leqslant\nu\leqslant\varepsilon$. We say that (a class) (d,ε) is a *vertex of* $\varXi_Y$ *with exponent* ε if $\varepsilon=0$ or ε is a characteristic Puiseux exponent of Y_d or there is $e\neq d$ such that $a_{d,\delta}=a_{e,\delta}$ if $\delta<\varepsilon$ and $a_{d,\varepsilon}\neq a_{e,\varepsilon}$. We call $(d,0)$ the *root* of $\varXi_Y$. We say that a vertex $w=(d,\delta)$ is a *descendent* of a vertex $v=(d,\varepsilon)$ $(w\gg v)$ if $\varepsilon<\delta$. We say that a vertex (d,ε) is a *shaft* if $a_{d,\varepsilon}=0$. By construction $b_v\geqslant 1$ for all v nonterminal.

For each $v=(d,\varepsilon)\in\mathcal{V}_Y$ let Y_v be the irreducible branch given by the Puiseux expansion $y=\varPhi_v(x)$, where $\varPhi_v(x)=\sum_{\nu<\delta}a_{d,\nu}x^\nu$, if v is nonterminal with a child $w=(d,\delta)$ and $\varPhi_v=\sum_{\nu\geqslant 0}a_{d,\nu}x^\nu$ otherwise.

For $R\gg 0$ let $\varSigma_\eta$ be the boundary of the polydisc $\{(x,y)\in\mathbb{C}^2\colon\ |x|\leqslant\eta,\ |y|\leqslant R\}$. Given $0<\eta\ll 1$ let $K_v=K_{v,\eta}=Y_v\cap\varSigma_\eta$ be the knot of the branch Y_v. Set $N_v=N_{v,\eta,\zeta_v}=\{(x,y)\colon\ |x|=\eta,\ |y-\varPhi_v(x)|\leqslant\zeta_v\}$, where $0<\zeta_v\ll 1$. The following lemma can be easily checked.

Lemma 1.1. *There are $\eta>0$ and closed neighborhoods N_v of K_v, $v\in\mathcal{V}_Y$, such that K_v is a deformation retract of N_v, $N_w\subset\operatorname{int}(N_v\setminus K_v)$ if $v\ll w$ and $N_v\cap N_w=\emptyset$ if $v\not\ll w$ and $w\not\ll v$.*

We fix a family $(N_v)_v$, where v runs over the set of vertices of $\varXi_Y$, in the conditions of the previous lemma and call it a *toric system for* Y. We set $L_\eta=\{\eta\}\times\mathbb{C}$ and $N_{v,\eta}=N_v\cap L_\eta$.

Let us associate to each vertex v a point $z_v\in\partial N_{v,\eta}$ and define paths connecting these points. Fix some $z_\phi\in\partial N_{\phi,\eta}$. Let v be a nonterminal vertex of $\varXi_Y$ to which it was associated $z_v\in\partial N_{v,\eta}$. Let D_v be the connected component of $N_{v,\eta}$ that contains z_v. Let c_v be the center of D_v. Given $z\in D_v$, $z\neq c_v$, let $r(z)$ be the radius of the disc D_v passing through z. Given $z,z'\in D_v$, $z\neq z'$, we say that $z<_v z'$ if z belongs to the line segment $[c_v,z']$ or $r(z')$ is placed to the right of $r(z)$, when we move along the boundary of D_v in the anticlockwise direction starting from z_v. We order the connected components of $(\bigcup_{w>v}N_w)\cap D_v$ accordingly to the order of the corresponding centers. For each $w>v$ denote by D_w the first connect component of $D_v\cap N_w$. If w is [not] a shaft let z_w be the point of ∂D_w whose distance to z_v $[\partial D_v]$ is minimum. Let $\tau_{w,v}$ be the path that starts at z_v, moves along ∂D_v in the anticlockwise direction until it reaches $r(z_w)$ and then moves inwards along this radius until it reaches z_w or a point of the boundary of some other disc. In this later case we move along this boundary in the clockwise direction until we reach again $r(z_w)$. In general, if $v\ll w$ we take $v_1,\dots,v_m$ s.t. $v=v_1<\dots<v_m=w$ and we set $\tau_{w,v}=\tau_{v_m,v_{m-1}}\cdots\tau_{v_2,v_1}$. The union B of all the paths $\tau_{w,v}$ is simply connected and is called the *base set*.

We call *standard meridian* and *standard parallel* of N_v a pair of positively oriented simple closed curves α_v and β_v defined over ∂N_v with base point z_v, such that α_v,β_v are homeomorphic to S^1, $\alpha_v\sim 0$, $\beta_v\sim K_v$ in $H_1(N_v)$, $\ell(\alpha_v,K_v)=1$ and $\ell(\beta_v,K_v)=0$. Here $\ell(\cdot,\cdot)$ denotes the linking number in the oriented homology 3-sphere $\varSigma_\eta$. It is well known that the standard meridian and the standard parallel are unique up to isotopy. In the sequel we still denote by α_v,β_v the standard meridian and the standard parallel of ∂N_v, with 'base point' B. If v is nonterminal, let $v_1,\dots,v_{b_v}$ be the children of v that are not shafts s.t. $D_{v_1}<_v\cdots<_v D_{v_{b_v}}$. There are positive integers μ_v,ν_v with $\gcd(\mu_v,\nu_v)=1$ s.t. $K_{v_i}\sim\mu_v\alpha_v+\nu_v\beta_v$ in $H_1(N_v\setminus K_v)$ for all i. Notice that μ_v and ν_v equal, respectively, the intersection multiplicity and the 'ramification index' of Y_v and Y_{v_i} (cf. [4]). Let r_v,s_v be integers such that $r_v\mu_v=s_v\nu_v+1$.

Theorem 1.2. *For each germ of plane curve Y, the local fundamental group of Y, $\pi_1(X\setminus Y,B)$ is presented by the generators α_v, β_v, v vertex of $\varXi_Y$, the relations $\beta_\phi=1$, $[\alpha_v,\beta_v]=1$ for all v and*

$$\alpha_w^{\nu_v\mu_v}\beta_w=\alpha_v^{\mu_v}\beta_v^{\nu_v},\quad \text{if } w>v \text{ and } w \text{ is not a shaft},\tag{1}$$

$$\alpha_w^{\mu_v}\beta_w^{\nu_v}=\alpha_v^{\mu_v}\beta_v^{\nu_v},\quad \text{if } w>v \text{ and } w \text{ is a shaft},\tag{2}$$

$$(\alpha_{v_1}\cdots\alpha_{v_{b_v}}\alpha_v^{s_v}\beta_v^{r_v})^{\nu_v}=(\alpha_v^{\mu_v}\beta_v^{\nu_v})^{r_v},\quad v \text{ nonterminal without shaft},\tag{3}$$

$$\alpha_{v_1}\cdots\alpha_{v_{b_v}}\alpha_v^{s_v}\beta_v^{r_v}=\alpha_{v_0}^{s_v}\beta_{v_0}^{r_v},\quad v \text{ nonterminal with shaft } v_0.\tag{4}$$

2. Proof of Theorem 1.2

In order to prove Theorem 1.2 consider a toric system for Y, $(N_v)_v$, $v \in \mathcal{V}_Y$. The group $\pi_1(X \setminus Y, B)$ is the quotient of $\pi_1(N_\phi \setminus \bigcup_w \operatorname{int} N_w, B)$ by the normal subgroup generated by β_ϕ, where w runs over the set of terminal vertices of Ξ_Y. A convenient decomposition of $N_\phi \setminus \bigcup_w \operatorname{int} N_w$ and an induction argument using van Kampen's theorem, reduces the computation of $\pi_1(N_\phi \setminus \bigcup_w \operatorname{int} N_w, B)$ to the computation of $\pi_1(N_v \setminus \cup_{w>v} \operatorname{int} N_v, z_v)$, where v runs over the set of nonterminal vertices of Ξ_Y. More precisely, if v is a nonterminal vertex of Ξ_Y with l children that are not shafts, $v_1, \ldots, v_l$, and a child that is a shaft v_0, it is enough to show that $\pi_1(N_v \setminus \bigcup_{i=0}^{l} \operatorname{int} N_{v_i}, z_v)$ is presented by $\alpha_v, \beta_v, \alpha_{v_i}, \beta_{v_i}$, $i = 0, \ldots, l$, s.t.

(a) $[\alpha_v, \beta_v] = [\alpha_{v_0}, \beta_{v_0}] = \cdots = [\alpha_{v_l}, \beta_{v_l}] = 1$,
(b) $\alpha_{v_1} \cdots \alpha_{v_l} \alpha_v^b \beta_v^a = \alpha_{v_0}^b \beta_{v_0}^a$ ($a, b \in \mathbb{Z}$ s.t. $a\mu_v = b\nu_v + 1$),
(c) $\alpha_v^{\mu_v} \beta_v^{\nu_v} = \alpha_{v_0}^{\mu_v} \beta_{v_0}^{\nu_v} = \alpha_{v_1}^{\nu_v \mu_v} \beta_{v_1} = \cdots = \alpha_{v_l}^{\nu_v \mu_v} \beta_{v_l}$.

The case where v does not have child that is a shaft, reduces to the previous one by adding the relation $\alpha_{v_0} = 1$ and by eliminating the generator β_{v_0} from the defining relations. Actually, we have $\alpha_{v_0} = (\alpha_v^{\mu_v} \beta_v^{\nu_v})^a (\alpha_{v_1} \cdots \alpha_{v_l} \alpha_v^b \beta_v^a)^{-\nu_v}$ and $\beta_{v_0} = (\alpha_v^{\mu_v} \beta_v^{\nu_v})^{-b} (\alpha_{v_1} \cdots \alpha_{v_l} \alpha_v^b \beta_v^a)^{\mu_v}$.

Let Y' be the curve with irreducible components given by the Puiseux expansion $y = \sum_{\varepsilon \in \mathbb{Q}^+} a'_{d,\varepsilon} x^\varepsilon$. Here $a'_{d,\varepsilon} = a_{d,\varepsilon}$ if (d, ε) is a vertex of Ξ_Y and $a'_{d,\varepsilon} = 0$ otherwise. We can replace Y by Y'.

Let $v = (d, \varepsilon)$ be a nonterminal vertex of Ξ_Y with l nonshaft children $v_1, \ldots, v_l$ and a child that is a shaft v_0. Set $n = \nu_v$ and set $p = o(v)$. We can rewrite the Puiseux expansion of the branch Y_v as

$$y = \sum_{i=1}^{p} b_i x^{m_1/n_1 + \cdots + m_i/(n_1 \cdots n_i)} = \sum_{i=1}^{p} b_i x^{\tilde{m}_i/(n_1 \cdots n_i)}, \tag{5}$$

where m_i, n_i, $i = 1, \ldots, p$, are nonnegative integers such that $m_i = 0$ if $b_i = 0$, $n_i = 1$ if $m_i = 0$ and $\gcd(m_i, n_i) = 1$ if $m_i \neq 0$. Since we are interested in studying the topology of $N_v \setminus \bigcup_{i=0}^{l} \operatorname{int} N_{v_i}$ we can assume that $b_i \neq 0$ for $i = 1, \ldots, p$. Set $\ell_1 = m_1$ and $\ell_j = m_j + n_j n_{j-1} \ell_{j-1}$ for $j = 2, \ldots, p+1$. By [2], Proposition 1A.1 (see also [4]) $\nu_u = n_{o(u)+1}$ and $\mu_u = \ell_{o(u)+1}$ for $u \ll v$ and $u = v$.

For each $j = 0, \ldots, l$, Y_{v_j} admits a Puiseux expansion of the form

$$y = \sum_{i=1}^{p} b_i x^{\tilde{m}_i/(n_1 \cdots n_i)} + c_j x^{\tilde{m}_{p+1}/(n_1 \cdots n_p n_{p+1})},$$

with $c_0 = 0$. Set $D^2 = \{z \in \mathbb{C}\colon |z| \leqslant 1\}$ and $S^1 = \partial D^2$. Set $\varphi_v(t) = \varphi_{v_0}(t) = \sum_{1 \leqslant i \leqslant p} b_i t^{\tilde{m}_i n_{i+1} \cdots n_p}$ and $\varphi_{v_j}(t) = \varphi_v(t^{n_{p+1}}) + c_j t^{\tilde{m}_{p+1}}$, $j = 1, \ldots, l$. Set $\tilde{\eta} = \eta^{1/(n_1 \cdots n_p)} > 0$. Consider $0 < \tilde{\zeta} \ll \zeta \ll 1$. We have parametrizations of N_v, and N_{v_j}, $j = 0, \ldots, l$, defined by $\psi_v(t, z) = (\eta t^{n_1 \cdots n_p}, \varphi_v(\tilde{\eta} t) + \zeta z)$, $\psi_{v_0}(t, z) = (\eta t^{n_1 \cdots n_p}, \varphi_{v_0}(\tilde{\eta} t) + \tilde{\zeta} z)$ and $\psi_{v_j}(t, z) = (\eta t^{n_1 \cdots n_{p+1}}, \varphi_{v_j}(\tilde{\eta}^{1/n_{p+1}} t) + \tilde{\zeta} z)$, for $j = 1, \ldots, l$, respectively. Here $t \in S^1$ and $z \in D^2$.

Let $r : \mathbb{C}^2_{\tilde{x},\tilde{y}} \to \mathbb{C}^2_{x,y}$ be the ramification defined by $r(\tilde{x}, \tilde{y}) = (\tilde{x}^{n_1 \cdots n_p}, \tilde{y} + \varphi_v(\tilde{x}))$. Let Z be the curve defined by $\prod_{j=0}^{l} (\tilde{y}^{n_{p+1}} - c_j \tilde{x}^{\tilde{m}_{p+1}}) = 0$. We have the trivial knots $\widetilde{K} = \widetilde{K}_0 = \{\tilde{y} = 0\} \cap \Sigma_{\tilde{\eta}}$ and torus knots of type $(\widetilde{m}_{p+1}, n_{p+1})$, $\widetilde{K}_j = \{\tilde{y}^{n_{p+1}} - c_j \tilde{x}^{\tilde{m}_{p+1}} = 0\} \cap \Sigma_{\tilde{\eta}}$, $j = 1, \ldots, l$. The corresponding tubular neighborhoods $\widetilde{N}$ and $\widetilde{N}_j$, $j = 0, \ldots, l$, parametrized, respectively, by $\tilde{\psi}(t, z) = (\tilde{\eta} t, \zeta z)$, $\tilde{\psi}_0(t, z) = (\tilde{\eta} t, \tilde{\zeta} z)$, $\tilde{\psi}_j(t, z) = (\tilde{\eta} t^{n_{p+1}}, c_j \tilde{\eta}^{\tilde{m}_{p+1}/n_{p+1}} t^{\tilde{m}_{p+1}} + \tilde{\zeta} z)$, $j \geqslant 1$, define a toric system for Z, where $t \in S^1$ and $z \in D^2$.

Since $\psi_v = r \circ \tilde{\psi}$ and $\psi_{v_j} = r \circ \tilde{\psi}_j$, $r : \widetilde{N} \xrightarrow{\sim} N$ defines an homeomorphism that maps $\widetilde{N}_j$ onto N_j for all j. Therefore $r_* : \pi_1(\widetilde{N} \setminus \bigcup_{j=0}^{l} \operatorname{int} \widetilde{N}_j, \tilde{z}) \xrightarrow{\sim} \pi_1(N_v \setminus \bigcup_{j=0}^{l} \operatorname{int} N_{v_j}, z_v)$, where $\tilde{z} \in \partial \widetilde{N}$ s.t. $r(\tilde{z}) = z_v$. Let $\tilde{\alpha}$, $\tilde{\beta}$ be, respectively, the standard meridian and the standard parallel of $\widetilde{N}$. Let $\tilde{\alpha}_j$, $\tilde{\beta}_j$ be, respectively, the standard meridian and the standard parallel of $\widetilde{N}_j$, $j = 0, \ldots, l$.

Lemma 2.1. *We have* $r_*(\tilde\alpha)=\alpha_v$, $r_*(\tilde\beta)=\alpha_v^{\ell_p n_p-\tilde m_p}\beta_v$, $r_*(\tilde\beta_0)=\alpha_{v_0}^{\ell_p n_p-\tilde m_p}\beta_{v_0}$, $r_*(\tilde\alpha_j)=\alpha_{v_j}$, $j=0,\dots,l$ *and* $r_*(\tilde\beta_j)=\alpha_{v_j}^{(\ell_{p+1}-\tilde m_{p+1})n_{p+1}}\beta_{v_j}$ *for* $j\geqslant 1$.

Proof. Clearly $r_*(\tilde\alpha)=\alpha_v$ and $r_*(\tilde\alpha_j)=\alpha_{v_j}$ for all j. By definition, $\widetilde K_j\sim\widetilde m_{p+1}\tilde\alpha+n_{p+1}\tilde\beta$ in $H_1(\widetilde N\setminus\widetilde K)$, $j\geqslant 1$. There are $s,t\in\mathbb Z$ s.t. $r_*(\widetilde K_j)\sim\widetilde m_{p+1}r_*(\tilde\alpha)+n_{p+1}r_*(\tilde\beta)=\widetilde m_{p+1}\alpha_v+n_{p+1}(s\,\alpha_v+t\,\beta_v)$, $j\geqslant 1$. Since $r_*(\widetilde K_j)=K_{v_j}\sim\ell_{p+1}\alpha_v+n_{p+1}\beta_v$, we get $t=1$ and $\ell_{p+1}=n_{p+1}s+\widetilde m_{p+1}$. Since $\ell_{p+1}=m_{p+1}+\ell_p n_p n_{p+1}$ and $\widetilde m_{p+1}=m_{p+1}+n_{p+1}\widetilde m_p$, $s=n_p\ell_p-\widetilde m_p$. The relation $r_*(\tilde\beta_0)=\alpha_{v_0}^{\ell_p n_p-\tilde m_p}\beta_{v_0}$ is proved in the same way. Now let $\tilde\sigma_j$ be the loop of $\partial\widetilde N_j$, $j=1,\dots,l$, obtained by moving the knot $\widetilde K_j$ directly away from $\widetilde K$. We can parametrize $\tilde\sigma_j$ by $\tilde\psi_j(t,c_j\tilde\eta^{\tilde m_{p+1}/n_{p+1}}t^{\tilde m_{p+1}})$, where $t\in S^1$. Since $\psi_{v_j}=r\circ\tilde\psi_j$, the loop $\sigma_{v_j}:=r_*(\tilde\sigma)$ is parametrized by $\psi_{v_j}(t,c_j\tilde\eta^{\tilde m_{p+1}/n_{p+1}}t^{\tilde m_{p+1}})$. Since $\varphi_{v_j}=\varphi_v(t^{n_{p+1}})+c_jt^{\tilde m_{p+1}}$, σ_{v_j} is the loop obtained by moving the knot K_{v_j} directly away from K_v. By the proof of Proposition 1A.1 of [2], $\tilde\sigma_j\sim\tilde\alpha_j^{\tilde m_{p+1}n_{p+1}}\tilde\beta_j$ and $\sigma_{v_j}\sim\alpha_j^{\ell_{p+1}n_{p+1}}\beta_{v_j}$. Thus $r_*(\tilde\beta_j)=\alpha_{v_j}^{(\ell_{p+1}-\tilde m_{p+1})n_{p+1}}\beta_{v_j}$, $j\geqslant 1$. □

By the previous lemma in order to prove Theorem 1.2 it is enough to show that $\pi_1(N_v\setminus\bigcup_{i=0}^l\operatorname{int}N_{v_i},z_v)$ is generated by $\alpha,\beta,\alpha_j,\beta_j$, $j=0,\dots,l$, verifying relations (a), (b) and (c), when $v=\phi$.

Let Y be the curve with irreducible components Y_d, $0\leqslant d\leqslant l$, where Y_d admits the Puiseux expansion $y=b_dx^{m/n}$, $0\leqslant d\leqslant l$, with $b_0=0$. Set $\mathbf e(t)=\exp(2\pi\sqrt{-1}\,t)$. We can assume that $b_d=\mathbf e((d-1)/(nl))$ for $d=1,\dots,l$. Let $N=N_{\phi,\eta,\zeta}$ and $N_d=N_{\phi_d,\eta,\tilde\zeta}$, $d=0,\dots,l$, be a toric system for Y.

We have a parametrization of $N\setminus\operatorname{int}N_0$ given by $X(t,s,\theta)=(\eta\,\mathbf e(nt),(\zeta(1-s)+\tilde\zeta s)\,\mathbf e(\theta+mt-1/(2nl)))$, $t,s,\theta\in[0,1]$. Set $T=T_{m,n,l}=X([0,1]^2\times\{j/(nl)\colon j=0,\dots,nl-1\})$. We call T a *turbine with shaft of type* (m,n,l). The turbine T is a retract by deformation of $N\setminus\bigcup_{d=0}^l\operatorname{int}N_d$. Take $r,s\in\mathbb Z$ s.t. $r\in\{0,\dots,n-1\}$ and $m=sn+r$. For $j\in\mathbb Z$ set $\gamma_j(t)=X(0,0,tj/(nl))$, $\tilde\gamma_j(t)=X(0,1,tj/(nl))$ and $\delta_j(t)=X(0,t,j/(nl))$, $t\in[0,1]$. Take (the homotopy classes of) the loops with base point $z=X(0,0,0)$, $\alpha=\gamma_{nl}$, $\alpha_0=\delta_0^{-1}\,\tilde\gamma_{nl}\,\delta_0$ and $\alpha_j=\gamma_{j-1}^{-1}\,\delta_{j-1}^{-1}\,\tilde\gamma_{j-1}\,\tilde\gamma_j^{-1}\,\delta_j\gamma_j$ for $j=1,\dots,nl$. Let β $[\beta_0]$ be the loop parametrized by $X(t,0,0)$ $[X(t,1,0)]$, $t\in[0,1]$. Clearly α,β are the standard generators of ∂N, α_0,β_0 are the standard generators of ∂N_0, and α_j, $j=1,\dots,l$ are the standard meridians of ∂N_j. The nl 'blades' of turbine T retract by deformation into the trajectories of $\omega_j(t)=X(t/n,1/2,j/(nl)+t(s+r/n))$, $t\in[0,1]$, $j=0,\dots,nl-1$. Set $\tilde\delta_j(t)=X(0,t/2,j/(nl))$, $t\in[0,1]$, and $\xi_j=\gamma_{j-1+rl}^{-1}\,\tilde\delta_{j-1+rl}^{-1}\,\omega_{j-1}\,\tilde\delta_{j-1}\,\gamma_{j-1}$, for all j.

Set $\theta_d=\alpha_1\alpha_2\cdots\alpha_d$ if $0\leqslant d\leqslant nl-1$ and set $\theta_d=\alpha_0^{-t}\,\theta_j\,\alpha^t$ if $d=tnl+j$ with $0\leqslant j\leqslant nl-1$.

Lemma 2.2. *The group* $\pi_1(T_{m,n,l},z)$ *is presented by the generators* $\alpha,\alpha_0,\dots,\alpha_{nl},\beta,\beta_0$ *and the relations*

(a$_1$) $[\alpha,\beta]=[\alpha_0,\beta_0]=1$,
(b$_1$) $\alpha=\alpha_0\cdots\alpha_{nl}$,
(c$_1$) $\alpha^s\beta=\theta_{rl+d}^{-1}\alpha_0^s\beta_0\theta_d$, $d\in\mathbb Z$.

Proof. Let T' be the union of $T\setminus\partial N_0$ with the trajectory of α_0. Let T'' be the union of $T\setminus\partial N$ with the trajectory of α. Let $\varphi:T'\cap T''\hookrightarrow T'$, $\psi:T'\cap T''\hookrightarrow T''$ be the inclusion maps. Remark that T' retracts by deformation into the union of ∂N with the trajectories of α_d, $1\leqslant d\leqslant nl$, T'' retracts by deformation into the union of ∂N_0 with the trajectories of α_d, $1\leqslant d\leqslant nl$, $T'\cap T''$ retracts by deformation into the connected graph that is the union of the trajectories of α_d's and the ξ_d's, $1\leqslant d\leqslant nl$. Hence the fundamental group of $T'\cap T''$ is the free group generated by α_0,α_d,ξ_d, $1\leqslant d\leqslant nl$,

$$\pi_1(T')=\langle\alpha,\beta,\alpha_0,\dots,\alpha_{nl}\colon\ [\alpha,\beta]=1,\ \alpha=\alpha_0\cdots\alpha_{nl}\rangle,$$
$$\pi_1(T'')=\langle\alpha,\beta,\alpha_0,\dots,\alpha_{nl},\beta_0\colon\ [\alpha_0,\beta_0]=1,\ \alpha=\alpha_0\cdots\alpha_{nl}\rangle,$$

$$\varphi_*\xi_d = \alpha^s\beta, \quad \psi_*\xi_d = \theta^{-1}_{rl+d-1}\alpha_0^s\beta_0\theta_{d-1}, \quad d = 1, \dots, (n-r)l, \tag{6}$$

$$\varphi_*\xi_d = \alpha^{s+1}\beta, \quad \psi_*\xi_d = \theta^{-1}_{d-1-(n-r)l}\alpha_0^{s+1}\beta_0\theta_{d-1}, \ \ d = (n-r)l+1, \dots, nl. \tag{7}$$

By van Kampen's theorem, $\pi_1(T_{m,n,l}, z)$ is generated by $\alpha, \alpha_0, \dots, \alpha_{nl}, \beta, \beta_0$ with relations (a_1), (b_1),

$$\alpha^s\beta = \theta^{-1}_{rl}\alpha_0^s\beta_0\theta_0 = \cdots = \theta^{-1}_{nl-1}\alpha_0^s\beta_0\theta_{(n-r)l-1}, \tag{8}$$

$$\alpha^{s+1}\beta = \theta_0^{-1}\alpha_0^{s+1}\beta_0\,\theta_{(n-r)l} = \cdots = \theta^{-1}_{rl-1}\alpha_0^{s+1}\beta_0\theta_{nl-1}. \tag{9}$$

Since $\alpha^{-1}\,\theta_d^{-1}\,\alpha_0 = \theta^{-1}_{d+nl}$, (8) and (9) are equivalent to (c_1). □

For $d = 1, \dots, l$ set $\sigma_d = \xi_{d+(n-1)rl}\cdots\xi_{d+rl}\xi_d$. By the proof of Proposition 1A.1 of [2] the standard parallel of $\partial N_j, \beta_j, j = 1, \dots, l$, is homotopic to $\alpha_j^{-nm}\sigma_j$. Hence it is enough to prove the following lemma.

Lemma 2.3. *$\pi_1(T_{m,n,l}, z)$ is generated by $\alpha, \beta, \alpha_0, \beta_0, \alpha_d, \sigma_d$, $d = 1, \dots, l$, verifying the relations*

(a_2) $[\alpha, \beta] = [\alpha_0, \beta_0] = [\alpha_d, \sigma_d] = 1$, *$d = 1, \dots, l$,*
(b_2) $\alpha_1\cdots\alpha_l = (\alpha_0^b\beta_0^a)(\alpha^b\beta^a)^{-1}$, *with $a, b \in \mathbb{Z}$ such that $am = bn + 1$,*
(c_2) $\sigma_d = \alpha_0^m\beta_0^n = \alpha^m\beta^n$, *$d = 1, \dots, l$.*

Proof. Since $\theta_j = \alpha_1\cdots\alpha_j$, $j = 0, \dots, nl$, $\pi_1(T_{m,n,l}, z)$ is generated by $\alpha, \beta, \alpha_0, \beta_0, \theta_1, \dots, \theta_{nl-1}$ with relations (a_1) and (c_1). Iterating t times the relation (c_1) one gets $\theta_j = (\alpha_0^s\beta_0)^{-t}\theta_{j+trl}(\alpha^s\beta)^t$. Since $m = sn + r$, $\theta_j = (\alpha_0^s\beta_0)^{-n}\alpha_0^{-r}\theta_j\alpha^r(\alpha^s\beta)^n = (\alpha_0^m\beta_0^n)^{-1}\theta_j(\alpha^m\beta^n)$, for $j \in \mathbb{Z}$. Taking into account the definition of the σ_d's and relations (6), (7), we have that $\sigma_d = \alpha^m\beta^n$, $d = 1, \dots, nl$, commutes with α_j, $j = 1, \dots, nl$. Since $ar = (b - as)n + 1$ and $\theta_{j+l+(b-as)nl} = \alpha_0^{as-b}\theta_{j+l}\alpha^{b-as}$, $\theta_j = (\alpha_0^b\beta_0^a)^{-1}\theta_{j+l}(\alpha^b\beta^a)$ and $\theta_{j+nl} = (\alpha_0^m\beta_0^n)^{-a}\theta_j(\alpha^m\beta^n)^a = \theta_j$. Hence $\pi_1(T_{m,n,l}, z)$ is generated by $\alpha, \beta, \alpha_0, \beta_0, \theta_1, \dots, \theta_l$ and verifies relations (a_2), (b_2) and (c_2).

Conversely, let us recover relations (b_1) and (c_1). Set $\theta_d = \alpha_1\cdots\alpha_d$, $0 \leqslant d \leqslant l$ and set $\theta_{d+tl} = (\alpha_0^b\beta_0^a)^t \times \theta_d(\alpha^b\beta^a)^{-t}$, $t \in \mathbb{Z}$. By (a_2) and (c_2) $\theta_{d+nl} = (\alpha_0^b\beta_0^a)^n\theta_d(\alpha^b\beta^a)^{-n} = \sigma_d^a\alpha_0^{-1}\theta_d\,\alpha\,\sigma_d^{-a} = \alpha_0^{-1}\theta_d\,\alpha$ and $\theta_{d+rl} = (\alpha_0^b\beta_0^a)^r\theta_d(\alpha^b\beta^a)^{-r} = \sigma_d^{b-as}\alpha_0^s\beta_0\,\theta_d\,(\alpha^s\beta)^{-1}\sigma_d^{as-b} = \alpha_0^s\beta_0\,\theta_d(\alpha^s\beta)^{-1}$. Moreover, we can replace the θ_j's by the α_j's. □

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