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Numerical Analysis

A dual finite element complex on the barycentric refinement

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Abstract

A simplicial mesh on an oriented two-dimensional surface gives rise to a complex $X^\bullet$ of finite element spaces centered on divergence conforming Raviart–Thomas vector fields and naturally isomorphic to the simplicial cochain complex. On the barycentric refinement of such a mesh, we construct finite element spaces forming a complex $Y^\bullet$, centered around curl conforming vector fields, naturally isomorphic to the simplicial chain complex on the original mesh and such that Y^{2-i} is in L^2 duality with X^i. In terms of differential forms this provides a finite element analogue of Hodge duality. ***To cite this article: A. Buffa, S.H. Christiansen, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Un complexe dual d'éléments finis sur le raffinement barycentrique. Un maillage simplicial sur une surface orientée bidimensionnelle donne lieu à un complexe $X^\bullet$ d'espaces d'éléments finis centré sur l'espace de Raviart–Thomas de champs de vecteurs à divergence conforme et naturellement isomorphe au complexe des cochaînes simpliciales. Sur le raffinement barycentrique d'un tel maillage, nous construisons des espaces d'éléments finis formant un complexe $Y^\bullet$, centré sur des champs de vecteurs à rotationnel conforme, naturellement isomorphe au complexe des chaînes simpliciales sur le maillage de départ et tel que Y^{2-i} soit en dualité L^2 avec X^i. En termes de formes différentielles, on obtient un analogue de la dualité de Hodge pour les éléments finis. ***Pour citer cet article : A. Buffa, S.H. Christiansen, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Let Γ be the boundary of a bounded domain in $\mathbb{R}^3$. We suppose that Γ is piecewise affine. Thus Γ is a two-dimensional orientable manifold. We chose the orientation on Γ induced by the outward pointing normal on Γ, which we denote by n. In fact all the results of this Note are valid in the slightly more general setting of an oriented piecewise linear manifold, but embedding it in $\mathbb{R}^3$ makes the presentation less technical and corresponds

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to the most important applications we have in mind, namely electromagnetic scattering problems [6]. For such problems, in particular in the context of integral equations, one encounters the problem finding finite dimensional spaces X^1 and Y^1 of vector fields such that $X^1 \subset \mathrm{H}_{\mathrm{div}}(\Gamma)$, $Y^1 \subset \mathrm{H}_{\mathrm{curl}}(\Gamma)$ and the L^2 duality is non-degenerate on $X^1 \times Y^1$. In this Note we let X^1 be the standard space of Raviart–Thomas vector fields and construct a new adequate space Y^1 (it should be remarked that setting $Y^1 = X^1 \times n$ does *not* provide an adequate answer, see [5]), guided by the use of differential complexes. It has applications to the preconditioning the electric field integral equation, providing an alternative to [5] that will be detailed elsewhere, and provides new discretization spaces for some of the formulations of impedance boundary conditions that can be found in e.g. [1].

2. Definitions

We equip Γ with a simplicial mesh denoted $\mathcal{T}_h$. Recall that a simplicial complex on Γ is a set of finite subsets (called simplexes) of Γ, such that: $\forall s \in \mathcal{T}_h\ \forall s' \subset s,\ s' \in \mathcal{T}_h$. For each integer i denote by $\mathcal{T}_h^i$ the subset of $\mathcal{T}_h$ consisting of i-dimensional simplexes (i.e. elements of cardinality $i+1$). For $s \in \mathcal{T}_h$ we denote by $|s|$ the convex envelope of s in $\mathbb{R}^3$, called the geometric realization of s. The elements of $\mathcal{T}_h^i$ are called vertexes when $i = 0$, edges when $i = 1$ and triangles when $i = 2$. We suppose that for each simplex s, $|s| \subset \Gamma$ and that the union of geometric realizations is Γ. Moreover we suppose that $\mathcal{T}_h$ is non-degenerate in the following sense: for each $s \in \mathcal{T}_h^i$, $|s|$ is indeed i-dimensional, and moreover $\forall s, s' \in \mathcal{T}_h$, $|s| \cap |s'| = |s \cap s'|$. Then $\mathcal{T}_h^i$ is empty for $i > 2$. The triangles of $\mathcal{T}_h$ are oriented by n, and for each edge an orientation is chosen. The oriented unit-norm tangent vector along an edge e is denoted τ_e.

On $\mathcal{T}_h$ we consider the lowest order finite-element complex (X_h^0, X_h^1, X_h^2) based on Raviart–Thomas divergence conforming vector fields (see [2] for details). It is defined by:

$$X_h^0 = \big\{u \in \mathrm{H}^1(\Gamma)\colon\ \forall t \in \mathcal{T}_h^2\ u|_{|t|} \in \mathrm{P}_1\big\}, \tag{1}$$

$$X_h^1 = \big\{u \in \mathrm{H}_{\mathrm{div}}(\Gamma)\colon\ \forall t \in \mathcal{T}_h^2\ u|_{|t|} \in \mathrm{RT}_0\big\}, \tag{2}$$

$$X_h^2 = \big\{u \in \mathrm{L}^2(\Gamma)\colon\ \forall t \in \mathcal{T}_h^2\ u|_{|t|} \in \mathrm{P}_0\big\}. \tag{3}$$

These spaces satisfy $\operatorname{curl} X_h^0 \subset X_h^1$ and $\operatorname{div} X_h^1 \subset X_h^2$, providing a complex $X_h^0 \to X_h^1 \to X_h^2$. We denote by $\lambda^i = (\lambda_s^i)$ the standard basis of X_h^i and $l^i = (l_s^i)$ the usual degrees of freedom (d.o.f.), both indexed by $s \in \mathcal{T}_h^i$. Explicitly l_v^0 is evaluation at the vertex v, l_e^1 is integration of the normal component along the edge e, and l_t^2 is integration on the triangle t. We have $l_s^i(\lambda_t^i) = \delta_{st}$.

The barycentric refinement of $\mathcal{T}_h$ is defined by dividing each triangle $s \in \mathcal{T}_h^2$, into six triangles by drawing the six edges joining the barycenter of s with the vertexes of s as well as the midpoints of its edges. The barycentric refinement of $\mathcal{T}_h$ is denoted $\mathcal{T}_h'$. In Fig. 1 the edges of $\mathcal{T}_h$ are drawn in bold, whereas non-bold segments are edges of $\mathcal{T}_h'$ (all bold segments are also edges of $\mathcal{T}_h'$). On $\mathcal{T}_h'$ we consider the slightly different finite-element complex $(\widetilde{X}_h^0, \widetilde{X}_h^1, \widetilde{X}_h^2)$ defined by:

$$\widetilde{X}_h^0 = \big\{u \in \mathrm{H}^1(\Gamma)\colon\ \forall t \in \mathcal{T}_h'^2\ u|_{|t|} \in \mathrm{P}_1\big\}, \tag{4}$$

$$\widetilde{X}_h^1 = \big\{u \in \mathrm{H}_{\mathrm{curl}}(\Gamma)\colon\ \forall t \in \mathcal{T}_h'^2\ u|_{|t|} \in \mathrm{RT}_0 \times n\big\}, \tag{5}$$

$$\widetilde{X}_h^2 = \big\{u \in \mathrm{L}^2(\Gamma)\colon\ \forall t \in \mathcal{T}_h'^2\ u|_{|t|} \in \mathrm{P}_0\big\}. \tag{6}$$

These spaces satisfy $\operatorname{grad} \widetilde{X}_h^0 \subset \widetilde{X}_h^1$ and $\operatorname{curl} \widetilde{X}_h^1 \subset \widetilde{X}_h^2$ so that we have a complex $\widetilde{X}_h^0 \to \widetilde{X}_h^1 \to \widetilde{X}_h^2$. Bases are constructed for these spaces on $\mathcal{T}_h'$ as for the previous spaces on $\mathcal{T}_h$ and denoted $(\tilde{\lambda}_s^i\colon\ s \in \mathcal{T}_h'^i)$.

The aim of this Note is to construct subspaces $Y_h^i \subset \widetilde{X}_h^i$ such that on the one hand L^2 duality on $Y_h^i \times X_h^{2-i}$ is non-degenerate and on the other hand they form a complex $Y_h^0 \to Y_h^1 \to Y_h^2$ (for the operators grad and curl respectively). We define these spaces by the construction of a basis and then check that our goals are fulfilled. For each $i \in \{0, 1, 2\}$ and each simplex $s \in \mathcal{T}_h^{2-i}$, let $\mu_s^i \in \widetilde{X}_h^i$ be the field attached to s constructed as a linear

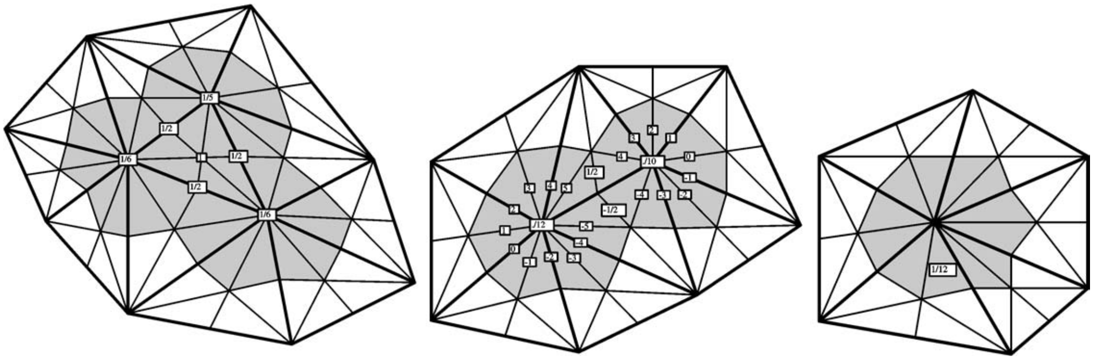

Fig. 1. Left: A basis element for Y_h^0 expressed in the basis of $\widetilde{X}_h^0$. Middle: A basis element for Y_h^1 expressed in the basis of $\widetilde{X}_h^1$ (edges are oriented away from the central one and given weight indicated at their origin). Right: A basis element for Y_h^1 expressed in the basis of $\widetilde{X}_h^2$ (all $2N$ triangles in the support should have the same coefficient $1/(2N)$).

combination of the functions $\tilde{\lambda}_t^i$ with the coefficients shown in Fig. 1 on the left for $i=0$, in the middle for $i=1$, and on the right for $i=2$. In each figure the shaded region is the support of the corresponding field. We then define Y_h^i by $Y_h^i = \operatorname{span}\{\mu_s^i\colon s \in \mathcal{T}_h^{2-i}\}$.

To construct degrees of freedom for these spaces we now fix some notations. For each triangle $t \in \mathcal{T}_h^2$ let t' denote its barycenter. For each edge $e \in \mathcal{T}_h^1$, let e' be union of (the geometric realizations of) the two edges of $\mathcal{T}_h'$ joining the barycenter of e to the barycenters of the two neighboring triangles. The tangent vector $\tau_{e'}$ along e' is oriented such that $\tau_{e'} \cdot (\tau_e \times n) > 0$. For each vertex $v \in \mathcal{T}_h^0$, denote by v' the union of (the geometric realizations of) the triangles of $\mathcal{T}_h'$ containing v. We have thus defined certain geometric objects s' on $\mathcal{T}_h'$ attached to simplexes $s \in \mathcal{T}_h$. We now define three families $M_h^i = (m_s^i)_{s \in \mathcal{T}_h^{2-i}}$ of d.o.f. by:

$$m_s^i : u \mapsto \int_{s'} u, \quad \text{with } \int_{s'} u = u(s')\ (i=0);\ = \int_{s'} u \cdot \tau_{e'}\ (i=1);\ = \int_{s'} u\ (i=2). \tag{7}$$

3. Properties

We first prove some rather algebraic properties. Straightforward checking gives:

Proposition 3.1. *For each $i \in \{0,1,2\}$ and each i-dimensional simplexes $s, t \in \mathcal{T}_h^i$ we have $m_s^i(\mu_t^i) = \delta_{st}$. In particular, for each i the family $\mu^i = (\mu_s^i)$ indexed by $s \in \mathcal{T}_h^{2-i}$ is a basis for Y_h^i, and a field $u \in Y_h^i$, $i = 0,1,2$, is uniquely determined the values $m_s^i(u)$ for $s \in \mathcal{T}_h^{2-i}$.*

Proposition 3.2. *The family of functions $(\mu_s^0\colon s \in \mathcal{T}_h^2)$ is a partition of unity.*

Proposition 3.3. *We have $\operatorname{grad} Y_h^0 \subset Y_h^1$ and $\operatorname{curl} Y_h^1 \subset Y_h^2$. Moreover the matrix of $\operatorname{grad}: Y_h^0 \to Y_h^1$ in the basis $\mu^0 \to \mu^1$ is the transpose of the matrix of $\operatorname{div}: X_h^1 \to X_h^2$ in the basis $\lambda^1 \to \lambda^2$, and similarly the matrix of $\operatorname{curl}: Y_h^1 \to Y_h^2$ in the basis $\mu^1 \to \mu^2$ is the transpose of the matrix of $\operatorname{curl}: X_h^0 \to X_h^1$ in the basis $\lambda^0 \to \lambda^1$.*

All these matrices are in fact incidence matrices, with entries in $\{-1,0,1\}$ according to the relative orientation.

For each $i \in \{0,1,2\}$ we denote by I_h^i the interpolation operator associated with the d.o.f. M_h^i. Explicitly I_h^i associates with any sufficiently regular field u (scalar or vector according to i) the element u_h of Y_h^i such that $\forall s \in \mathcal{T}_h^{2-i}$ $m_s^i(u_h) = m_s^i(u)$. Applying Stokes theorem on the geometric elements s' we associated with the simplexes $s \in \mathcal{T}_h$ in order to define the d.o.f. gives:

Proposition 3.4. *Let $\Omega^0 \subset \mathrm{H}^1(\Gamma)$, $\Omega^1 \subset \mathrm{H}_{\mathrm{curl}}(\Gamma)$ and $\Omega^2 \subset \mathrm{L}^2(\Gamma)$ be the subspaces consisting of piecewise smooth fields. The interpolators satisfy the following commuting diagram*:

$$\begin{tikzcd} \Omega^0 \arrow[r, "\mathrm{grad}"] \arrow[d, "I_h^0"'] & \Omega^1 \arrow[r, "\mathrm{curl}"] \arrow[d, "I_h^1"'] & \Omega^2 \arrow[d, "I_h^2"'] \\ Y_h^0 \arrow[r, "\mathrm{grad}"] & Y_h^1 \arrow[r, "\mathrm{curl}"] & Y_h^2 \end{tikzcd} \tag{8}$$

The Poincaré duality on $X^\bullet$ (or an elementary argument based on the Euler–Poincaré formula) gives:

Proposition 3.5. *In the complex $Y^\bullet$, the cohomology groups have the same dimension as in $X^\bullet$.*

We now turn to metric properties of the spaces Y_h^i and focus on norms relative to traces (see [3,4]). Put $Y^0 = \mathrm{H}^{1/2}(\Gamma)$, $Y^1 = \mathrm{H}_{\mathrm{curl}}^{-1/2}(\Gamma)$ and $Y^2 = \mathrm{H}^{-1/2}(\Gamma)$. We suppose that we have a family of meshes $\mathcal{T}_h$ indexed by the mesh-width h and let $h \to 0$. Recall that a family (Z_h) of subspaces of a normed space Z is called approximating if $\forall u \in Z \ \lim_{h\to 0} \inf\{\|u_h - u\|\colon u_h \in Z_h\} = 0$. We suppose that the family of triangulations is quasiuniform. The most basic result is:

Proposition 3.6. *For each $i \in \{0, 1, 2\}$, the family (Y_h^i) is approximating in Y^i.*

Various convergence orders can also be obtained but are more technical than for standard finite elements. We will come back to this elsewhere. We now show that the L^2-dualities on $Y_h^i \times X_h^{2-i}$ are non-degenerate. Put $X^0 = \mathrm{H}^{1/2}(\Gamma)$, $X^1 = \mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)$, $X^2 = \mathrm{H}^{-1/2}(\Gamma)$.

Proposition 3.7. *For each $i \in \{0, 1, 2\}$, there is $\bar{h} > 0$ and C such that for all $h < \bar{h}$ we have*

$$\inf_{u\in X_h^i} \sup_{v\in Y_h^{2-i}} \frac{\int uv}{\|u\|_{X^i}\|v\|_{Y^{2-i}}} \geqslant 1/C. \tag{9}$$

Finally we remark that the results of this Note can be expressed in a very natural way in terms of differential forms. In this language the dualities we construct are Hodge dualities for finite element spaces. The construction can also be extended to the case of surfaces with a boundary (where one looks for spaces dual to the standard finite element spaces satisfying homogeneous boundary condition). Three (and higher) dimensional analogues are under investigation.

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