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Group Theory

Superrigidity for irreducible lattices and geometric splitting

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Abstract

We propose general superrigidity results for actions of irreducible lattices on $CAT(0)$ spaces. In particular, we obtain a new and self-contained proof of Margulis' superrigidity theorem for uniform irreducible lattices in non-simple groups. However, the statements hold for lattices in products of arbitrary groups; likewise, the geometric representations need not be linear. The proof uses notably a new splitting theorem which can be viewed as an infinite-dimensional and singular generalization of the Lawson–Yau/Gromoll–Wolf theorem. *To cite this article: N. Monod, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

La super-rigidité des réseaux irréductibles et un théorème de décomposition. Nous exposons des résultats de super-rigidité pour les actions de réseaux irréductibles en géométrie de Hadamard, singulière ou non. Une de nos motivations est de présenter une preuve élémentaire du théorème de super-rigidité de Margulis pour les réseaux uniformes dans les groupes algébriques semi-simples (non simples); nos méthodes s'appliquent cependant aux réseaux dans des produits de groupes complètement généraux. Notre preuve repose notamment sur un théorème de décomposition qui généralise le théorème de Lawson–Yau/Gromoll–Wolf aux dimensions infinies, ou plus précisément aux espaces $CAT(0)$ complets généraux. *Pour citer cet article : N. Monod, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Le théorème de super-rigidité de Margulis s'énonce comme suit dans le cas de groupes semi-simples (non simples) généraux [6] :

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Théorème (Margulis). Soit Γ un réseau irréductible dans un produit fini $G = \prod_{\alpha \in A} \mathbf{G}_{\alpha}(k_{\alpha})$, où chaque \mathbf{G}_{α} est un k_{α} -groupe semi-simple connexe simplement connexe sans facteurs k_{α} -anisotropes, k_{α} un corps local, $|A| \geq 2$. Soient k un corps local, \mathbf{H} un k -groupe connexe adjoint k -simple et $\tau : \Gamma \rightarrow \mathbf{H}(k)$ un homomorphisme d'image Zariski-dense et non bornée.

Alors τ s'étend en un homomorphisme continu $\tilde{\tau} : G \rightarrow \mathbf{H}(k)$.

Nous proposons dans cette Note de nous débarrasser complètement du contexte des groupes algébriques ou de Lie et de formuler une généralisation du théorème ci-dessus (principalement dans le cas des réseaux co-compacts) dans le contexte suivant. (i) Concernant G , on ne gardera que sa structure produit : nous supposons donc que $G = \prod G_i$ soit un produit de $2 \leq n < \infty$ groupes localement compacts σ -compacts. (ii) De $\mathbf{H}(k)$, on ne retiendra que l'aspect de la courbure négative : rappelons en effet que $\mathbf{H}(k)$ se réalise essentiellement comme groupe d'isométries de l'espace symétrique ou de l'immeuble de Bruhat–Tits associé, qui sont deux exemples d'espaces métriques CAT(0) ; nous considérerons donc un groupe d'isométries H d'un espace CAT(0). (iii) Enfin, Γ sera dit *irréductible* si ses projections dans chaque G_i sont denses.

Un des avantages de cette généralité est qu'elle nous contraint à trouver des preuves élémentaires ; ainsi, nous obtenons en particulier une nouvelle démonstration du théorème de Margulis dans le cadre ci-dessus. Nos méthodes utilisent la géométrie CAT(0) et sont donc insensibles aux distinctions entre les cas archimédiens ou non, et indifférentes aux arcanes des caractéristiques positives.

Soit donc Γ un réseau co-compact irréductible dans un produit $G = \prod G_i$ de groupes quelconques, localement compacts σ -compacts.

Théorème 0.1. Soit X un espace CAT(0) propre avec une Γ -action par isométries sans point fixe global. Alors il existe une partie fermée non vide Γ -invariante $\mathcal{C} \subseteq \partial X$ sur laquelle la Γ -action s'étend continûment en une G -action.

Pour obtenir une formulation qui comprenne le cadre de Margulis, nous devons trouver un remplacement pour la notion de Zariski-densité dans un groupe k -simple. Nous dirons donc qu'un groupe L d'isométries d'un espace CAT(0) propre X est *indécomposable* si pour toute partie $\mathcal{C} \subseteq \partial X$ fermée non vide L -invariante, le fixateur de \mathcal{C} dans $\text{Isom}(X)$ est trivial et son stabilisateur est fermé pour la topologie de la convergence simple (ce critère est satisfait dans le cas algébrique).

Corollaire 0.2. Soit $H < \text{Isom}(X)$ un sous-groupe fermé. Alors tout homomorphisme $\tau : \Gamma \rightarrow H$ d'image indécomposable et non bornée s'étend en un homomorphisme continu $\tilde{\tau} : G \rightarrow H$.

Nous indiquons dans la partie anglaise comment généraliser aux espaces CAT(0) qui ne sont pas localement compacts ; on présente aussi une dichotomie « arithméticité/non linéarité ».

Notre preuve consiste à étudier un G -espace CAT(0) induit Y d'applications $G/\Gamma \rightarrow X$; nous nous intéressons donc à présent à une action par isométries d'un produit $G = \prod G_i$ de groupes topologiques tout à fait quelconques sur un espace CAT(0) complet Y . Notons que même si X est des plus simples, Y sera de dimension infinie.

Dans ce cadre, le bord à l'infini ∂Y n'est guère utile ; nous dirons plutôt que la G -action est *évanescence* s'il existe une partie non bornée $T \subseteq Y$ telle que pour tout compact $Q \subseteq G$ l'ensemble $\{d(gy, y) : g \in Q, y \in T\}$ est borné.

Théorème 0.3. Si la G -action n'est pas évanescence, alors il existe une partie fermée convexe non vide canonique $Z \subseteq Y$ qui se décompose G -isométriquement $Z \cong \prod Z_i$ en produit de G_i -espaces Z_i .

1. Superrigidity

Mostow’s theorem on the rigidity of locally symmetric spaces of finite volume can be rephrased in the language of groups: Isomorphisms between irreducible lattices in suitable semisimple Lie groups extend to isomorphisms between the Lie groups. In the early 1970s, Margulis undertook to prove a very general family of rigidity theorems by considering homomorphisms, i.e. linear representations, in lieu of the above isomorphisms. He established indeed that, in higher rank, homomorphisms to semisimple groups over arbitrary local fields extend to continuous homomorphisms of the ambient groups; a paramount motivation for considering also non-Archimedean local fields is that Margulis could then deduce a complete classification of higher rank lattices (arithmeticity [5,6]).

In his 1988 paper [10], Venkataramana included the positive characteristic case. Margulis’ superrigidity theorem can be found in its full generality in [6]; here is how it reads for semisimple non-simple groups:

Theorem 1.1 (Margulis). *Let Γ be an irreducible lattice in $G = \prod_{\alpha \in A} \mathbf{G}_{\alpha}(k_{\alpha})$, where $2 \leq |A| < \infty$, k_{α} are local fields, and \mathbf{G}_{α} are connected simply connected semi-simple k_{α} -groups without k_{α} -anisotropic factors. Let k be a local field, \mathbf{H} a connected adjoint k -simple k -group and $\tau : \Gamma \rightarrow \mathbf{H}(k)$ a homomorphism with Zariski-dense unbounded image.*

Then τ extends to a continuous homomorphism $\tilde{\tau} : G \rightarrow \mathbf{H}(k)$.

The purpose of this Note is to announce a superrigidity theorem for irreducible lattices (mostly uniform) in products of arbitrary locally compact groups [7]. The homomorphisms we consider range in isometry groups of CAT(0) spaces. Since semisimple Lie/algebraic groups are (essentially) the isometry groups of the corresponding symmetric spaces or Bruhat–Tits buildings, this setting subsumes the case of linear representations over arbitrary fields as in Theorem 1.1.

We shall say that a lattice Γ in a product $G = G_1 \times \dots \times G_n$ of arbitrary locally compact groups is *irreducible* if the projection of Γ to each factor G_i is dense. In the classical semi-simple case, this follows essentially from the stronger notion of algebraic irreducibility. We may always reduce to the irreducible case: Indeed, after replacing each G_i with the closure of the projection of Γ , the group Γ is an irreducible lattice in the resulting product.

The Zariski-density is a necessary assumption for the above formulation of Theorem 1.1, and we shall propose two geometric replacements for it. But first, we propose a statement which can be made without any further assumption upon passing to the boundary at infinity:

Theorem 1.2. *Let Γ be an irreducible uniform lattice in a product $G = G_1 \times \dots \times G_n$ of locally compact σ -compact groups. Let Γ act by isometries on a proper CAT(0) space X without global fixed point.*

Then there is a non-empty closed Γ -invariant set $\mathcal{C} \subseteq \partial X$ on which the Γ -action extends continuously to a G -action. Moreover this action factors through $G \rightarrow G_i$ for some $i = 1, \dots, n$.

We now propose our first replacement for Zariski-density.

Definition 1.3. Let X be a proper CAT(0) space. A subgroup L of $\text{Isom}(X)$ is *indecomposable* if for every non-empty L -invariant closed subset $\mathcal{C} \subseteq \partial X$, its fixator (=pointwise stabiliser) in $\text{Isom}(X)$ is trivial and its stabiliser in $\text{Isom}(X)$ is closed for the topology of pointwise convergence in \mathcal{C} .

This always holds in the setting of Theorem 1.1. The essence of our definition is to exclude fixed points at infinity or product decompositions of the target space.

Corollary 1.4. *Let Γ be an irreducible uniform lattice in a product $G = G_1 \times \dots \times G_n$ of locally compact σ -compact groups. Let $H < \text{Isom}(X)$ be a closed subgroup, where X is a proper CAT(0) space, and let $\tau : \Gamma \rightarrow H$ be a homomorphism with indecomposable unbounded image.*

Then τ extends to a continuous homomorphism $\tilde{\tau} : G \rightarrow H$.

This corollary immediately implies Margulis' Theorem 1.1 for uniform lattices.

We point out that our approach is so simple-minded that it erases any distinction between Lie groups and non-Archimedean groups; in particular, the idiosyncrasies of positive characteristic become irrelevant.

Let us comment on two settings of intermediate generality: (i) If we keep $H = \mathbf{H}(k)$, the corollary shows that for an irreducible uniform lattice in a general product group G , all completely reducible linear representations over all local fields are completely determined by the continuous linear representations of G . (ii) When G is an algebraic group, the corollary still presents a new family of superrigidity results.

Our second replacement for Zariski-density is tailored to suit the needs of infinite-dimensional geometry, where the boundary is not always a satisfactory tool.

Definition 1.5. Let X be any CAT(0) space and $L < \text{Isom}(X)$ a group of isometries. We call L *reduced* if X has no Euclidean factor and there is no unbounded closed convex subset $Y \subsetneq X$ such that for all $\ell \in L$, ℓY is at finite distance from Y .

Theorem 1.6. Let Γ be an irreducible uniform lattice in a product $G = G_1 \times \cdots \times G_n$ of locally compact σ -compact groups. Let $H < \text{Isom}(X)$ be a closed subgroup, where X is any complete CAT(0) space, and let $\tau : \Gamma \rightarrow H$ be a homomorphism with reduced unbounded image.

Then τ extends to a continuous homomorphism $\tilde{\tau} : G \rightarrow H$.

The above results are stated for uniform (i.e. cocompact) lattices. Nevertheless, our proofs apply for certain non-uniform lattices, including the arithmetic case and most Kac–Moody lattices. The methods also produce an elementary proof of Margulis' commensurator superrigidity.

2. Splitting

In the proof of our superrigidity results, we need a splitting theorem for CAT(0) spaces that are not locally compact – regardless of whether the superrigidity theorem itself is about a locally compact space or not. In that setting, we need a replacement for the notion of fixed points at infinity:

Definition 2.1. Let G be a topological group with a continuous action by isometries on a metric space Y . The G -action on Y is *evanescent* if there is an unbounded set $T \subseteq Y$ such that for every compact set $Q \subseteq G$ the set $\{d(gy, y) : g \in Q, y \in T\}$ is bounded.

When Y is proper CAT(0), evanescence is simply equivalent to fixing a point at infinity in ∂Y ; however the following splitting theorem fails if rephrased naively in terms of ∂Y . Observe that there is no assumption whatsoever on the topology of G or Y :

Theorem 2.2. Let Y be a complete CAT(0) space and $G = G_1 \times \cdots \times G_n$ any product of arbitrary topological groups with a non-evanescent continuous G -action by isometries on Y .

Then there is a canonical non-empty closed convex G -invariant subspace $Z \subseteq Y$ which splits G -equivariantly isometrically as a product $Z_1 \times \cdots \times Z_n$ of G_i -spaces Z_i .

The special case where Y is locally compact is not useful to us, but we point out that it improves slightly on the known generalisations [9,1,3] of the Lawson–Yau/Gromoll–Wolf theorem [4,2]:

Corollary 2.3. Let Y be a proper CAT(0) space with a G -action by isometries, where $G = G_1 \times \cdots \times G_n$ is any product of groups G_i .

Either there is a G -fixed point in ∂Y , or there is a non-empty closed convex G -invariant subspace $Z \subseteq Y$ which splits G -equivariantly isometrically as a product of G_i -spaces Z_i .

3. Arithmeticity versus non-linearity

Following Margulis, superrigidity statements lead to arithmeticity results. It is therefore particularly interesting to apply our superrigidity to groups that are *not* arithmetic.

Let $\Gamma < G$ be an irreducible uniform lattice, where $G = G_1 \times \dots \times G_n$ is a product of topologically simple compactly generated locally compact groups. (In fact, we can also prove Theorem 3.1 below for groups G_i that are far from topologically simple – for instance, residually finite – provided they have *few factors* in a precise sense.) We assume that the projection of Γ to any (proper) subproduct of G_i 's is non-discrete; this still follows from the algebraic notion of irreducibility (after factoring out the compact factors). We then have the following alternative [8].

Theorem 3.1. *Either:*

- (i) *There is a topological isomorphism $G \cong \prod_{v \in S} \mathbf{H}(K_v)^+$ under which Γ is commensurable with $\mathbf{H}(K(S))$, where K is a global field, \mathbf{H} a connected absolutely simple adjoint K -group, S a finite set of inequivalent valuations; or:*
- (ii) *Any homomorphism from Γ to any linear group over any field of characteristic $\neq 2, 3$ has finite image.*

As a curious by-product of the proofs, there cannot be any lattice as above in ‘mixed’ products of a semi-simple Lie/algebraic group and a topological group without non-compact linear factors.

4. About the proofs

4.1. Superrigidity

Let $\Gamma < G = \prod G_i$ be an irreducible uniform lattice. To any CAT(0) Γ -space X we associate a CAT(0) G -space Y as follows. We consider right Γ -equivariant measurable maps $f: G \rightarrow X$ that are L^2 in the sense that the distance to any point in X is square summable on (a fundamental domain for) G/Γ . Defining the distance between $f, f' \in Y$ by $d^2(f, f') = \int_{G/\Gamma} d^2(f(g), f'(g)) dg$ yields a CAT(0) space upon passing to function classes; G acts isometrically by left multiplication. Notice that Y is in general not proper: For instance, any line in X yields a Hilbert space $L^2(G/\Gamma)$ in Y .

The G -action on Y reflects the properties of the Γ -action on X ; for instance, it is immediate that X has a Γ -fixed point if and only if Y has a G -fixed point. We have to deal with the more delicate question of evanescence. In our setting, we may assume that the Γ -action on X is non-evanescent; we prove that the G -action on Y is also non-evanescent. To this end we show that if $T \subseteq Y$ were a set as in Definition 2.1, then T consists of functions $f \in Y$ whose image is uniformly approximately bounded on compact sets. Therefore, replacing $f \in T$ with points in X is meaningful as far as evanescence is concerned, and will indeed yield evanescence of the Γ -action on X , contrary to our assumptions. (The actual choice of a point for f is irrelevant; we chose to use barycentres.)

At this point we may apply Theorem 2.2 and obtain a splitting $Y \supseteq Z = Z_1 \times \dots \times Z_n$ into G_i -spaces. We shall now focus on Z_i viewed as subset of Y . For any $f, f' \in Z_i$ and any $h \in G_j$ with $j \neq i$, we have a Euclidean rectangle $\{f, f', hf, hf'\}$ because of the splitting. It follows from a convexity argument that for a.e. $g \in G$ the points $\{f(g), f'(g), (hf)(g), (hf')(g)\}$ still form a ‘parallelogram’ in X ; that is, $d(f(g), f'(g)) =$

$d(f(h^{-1}g), f'(h^{-1}g))$ for almost all $g \in G$. On the other hand, the collection of those $h \in G_{j \neq i}$ acts ergodically on G/Γ since the projection of Γ to G_i is dense; this implies that $d(f(g), f'(g))$ is a.e. constant.

If we could evaluate f at the identity, the map $Z_i \rightarrow X, f \mapsto f(e)$ would therefore be a Γ -equivariant isometric map. Applying the argument to each factor Z_i , we see that Z consists of continuous functions, and the evaluation maps essentially finish the proof. For instance, for Theorem 1.2, observe that the boundary ∂Z_i is G -invariant; therefore, since the isometric Γ -map $Z_i \rightarrow X$ extends to a Γ -homeomorphism of ∂Z_i onto its image $\mathcal{C} \subseteq \partial X$, it extends continuously the Γ -action on \mathcal{C} to G .

4.2. Splitting

Let $G = \prod_{i=1}^n G_i$ be a product of n topological groups with a non-evanescent continuous action by isometries on a complete CAT(0) space Y ; we may reduce to $n = 2$. We seek to analyse the minimal non-empty convex G_1 -subspaces of Y , imitating [9] and [1, II.6.21]. Indeed, a well-known convexity argument (‘Sandwich Lemma’) shows that any two such subspaces $C, C' \subseteq Y$ span a product $C \times [0, d] \subseteq Y$. There are two main points to address: (i) Existence; (ii) The foliation by such $C \subseteq Y$ has a *pairwise* product structure only, whilst we desire a global isometric equivariant splitting.

For (i), we first observe that the strict convexity of Y provides for a geometric analogue of the Banach–Alaoglu theorem: Any intersection of a directed family of non-empty closed bounded convex sets is non-empty. This is then played off against non-evanescence to yield existence.

For (ii), the difficulty is partly that subspaces of Y lack any reasonable notion of geodesic extension (which would be a replacement for a Riemannian structure). It turns out however that the *failure* of a global isometric splitting would provide a non-trivial holonomy; it follows from minimality that the holonomy consists of Clifford translations, which have an axis – thus providing us with enough geodesic lines to rule them out.

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