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Differential Geometry

Non-unimodular Lie foliations

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Abstract

Let $\mathcal{F}$ be a G-Lie foliation on a compact manifold M. If $\mathcal{F}$ is not unimodular then either M or the closures of the leaves fiber over S^1. ***To cite this article: E. Macias-Virgós, P. Martín-Méndez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Feuilletages de Lie non unimodulaires. Soit $\mathcal{F}$ un G-feuilletage de Lie sur une variété compacte M. Si $\mathcal{F}$ n'est pas unimodulaire alors ou bien M ou bien les adhérences des feuilles fibrent sur S^1. ***Pour citer cet article : E. Macias-Virgós, P. Martín-Méndez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Soit M une variété compacte, G un groupe de Lie connexe simplement connexe. Un G-feuilletage de Lie sur M [8] est défini par une 1-forme de rang maximal sur M à valeurs dans l'algèbre de Lie de G, telle que $d\omega = (-1/2)[\omega, \omega]$. Le feuilletage $\mathcal{F}$ est *unimodulaire* si sa cohomologie basique vérifie la dualité de Poincaré. Rappelons que la cohomologie basique $H(M/\mathcal{F})$ d'une variété feuilletée est celle du complexe des formes basiques, c'est-à-dire des formes différentielles α sur M qui vérifient $i_X\alpha = 0$ et $i_X\, d\alpha = 0$ pour tout champ de vecteurs X tangent à $\mathcal{F}$.

Carrière [1] a donné le premier exemple de feuilletage de Lie non unimodulaire. Plus tard Masa [7] a montré que l'unimodulairité d'un feuilletage riemannien (et a fortiori de Lie) équivaut a l'existence sur la variété d'une métrique riemannienne pour laquelle les feuilles sont des sous-variétés minimales. Dans l'exemple de Carrière,

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la variété $M = T_A^3$ fibre sur le cercle. Nous nous proposons de démontrer le résultat général suivant : si un G-feuilletage de Lie (avec ou sans feuilles denses) n'est pas unimodulaire alors ou bien la variété M ou bien les adhérences des feuilles fibrent sur S^1.

Les ingrédients essentiels de la preuve sont le théorème de Tischler [9] et le résultat suivant obtenu par El Kacimi et Nicolau [3] : un G-feuilletage de Lie est unimodulaire si et seulement si G et l'adhérence K du groupe d'holonomie (qui n'est pas connexe en général) sont des groupes de Lie unimodulaires.

Il faut rappeler que l'unimodulairité d'un groupe de Lie *connexe* équivaut à celle de son algèbre de Lie, mais ceci n'est pas le cas pour le sous-groupe K.

1. Introduction

Let M be a compact manifold and G a connected simply connected Lie group. A G-Lie foliation on M [8] is that defined by a maximal rank 1-form ω on M with values in the Lie algebra of G, such that $\mathrm{d}\omega = (-1/2)[\omega, \omega]$. The foliation $\mathcal{F}$ is said to be *unimodular* if the basic cohomology verifies Poincaré duality. Recall that the basic cohomology $H(M/\mathcal{F})$ of a foliated manifold is that of the complex of basic forms, i.e. differential forms α on M verifying $i_X\alpha = 0$ and $i_X\,\mathrm{d}\alpha = 0$ for every vector field X tangent to $\mathcal{F}$.

The first example of a non unimodular Lie foliation was given by Carrière in [1]. Later Masa [7] proved the long-standing conjecture that unimodularity of a Riemannian foliation is equivalent to the existence of a Riemannian metric for which the leaves are minimal submanifolds.

In Carrière's example, the ambient manifold $M = T_A^3$ fibers over S^1. The aim of this paper is to prove the following general property of non-unimodular Lie foliations, even when the leaves are not dense.

Theorem 1.1. *If a G-Lie foliation on a compact manifold M is not unimodular then either M or the closures of the leaves fiber over S^1.*

2. Non-unimodular Lie foliations

The following result about the structure of Lie foliations is due to Fédida [4]. The G-Lie foliation $\mathcal{F}$ is completely determined by the holonomy morphism $h : \pi_1(M) \to G$ and the developing map $D : \widetilde{M} \to G$, where $\widetilde{M}$ is the covering space of M associated to the kernel of h. The map D is an h-equivariant fiber bundle. Let Γ be the image of h, and K its closure in G. Let K_e be the connected component of the identity in K. When the leaves of $\mathcal{F}$ are not dense in M (that is $K \neq G$), then the closures of the leaves are the fibers of a fiber bundle $M \to W$ over the basic manifold $W = G/K$. In this case $\mathcal{F}$ induces on each fiber another Lie foliation modeled by (the universal covering of) K_e (for the usual notations see also [6]).

It follows that the basic cohomology $H(M/\mathcal{F})$ is isomorphic to $H_K(G)$, the De Rham cohomology of differential forms on G invariant by K. It is finite dimensional [2]; hence duality is equivalent to $H_K^n(G) \neq 0$, $n = \operatorname{codim} \mathcal{F}$. We shall exploit the following result from El Kacimi and Nicolau.

Theorem 2.1 [3]. *$H_K^n(G) \neq 0$ if and only if the Lie groups G and K are unimodular.*

Recall that the modular function of the Lie group G is given by $m_G(x) = \det \mathrm{Ad}_G(x)$. Due to connectedness, the unimodularity of G (analogously K_e) is equivalent to that of its Lie algebra (i.e. $\operatorname{trace} \mathrm{ad}_X = 0$), but this is not the case for K.

Proof of Theorem 1.1. Let the foliation $\mathcal{F}$ be non-unimodular. If the Lie group G is not unimodular, we can consider the non-trivial multiplicative modular map $m_G : G \to \mathbb{R}^+$. Then the compositions $\log m_G \circ D$ and

$\log m_G \circ h$ can be viewed respectively as a developing map and a holonomy morphism defining a codimension one Lie foliation on M. In other words we have a foliation defined by a closed form. By Tischler's theorem [9], M fibers over S^1.

On the other hand, if G is unimodular, K can not be unimodular by Theorem 2.1, while K_e may be unimodular or not (see Remark 1 below).

(i) If K_e is not unimodular, by taking into account its modular function the argument above shows that the closures of the leaves (where there is an induced $\widetilde{K}_e$-Lie foliation) fiber over S^1.

(ii) If K_e is unimodular, let m_K be the non-trivial multiplicative map $K \to \mathbb{R}^+$ given by $m_K(x) = |\det Ad_K(x)|$. We shall prove the existence of a map $m : G \to \mathbb{R}^+$ such that $m_{|K} = m_K$ and $m(xy) = m(x) \cdot m(y)$ for all $x \in G$, $y \in K$. Then, by considering the compositions $f \circ D$ and $f \circ h$, where $f = \log m$, Tischler's theorem allows us again to ensure that M fibers over S^1. Notice that the map $f : G \to \mathbb{R}$ is onto, because the map $m_{|K} = m_K \neq 1$ is not bounded, and the Lie group G is connected. The equivariance follows from the properties of m.

Let us show the existence of the map m. Let $W = G/K$ be the basic manifold of the foliation. Its universal covering is $\widetilde{W} = G/K_e$ and $\pi_1(W) = K/K_e$. Since $m_K(x) = 1$ for $x \in K_e$, the morphism

$$\overline{m}_K : K/K_e \to \mathbb{R}^+$$

is well defined. Then

$$\log \overline{m}_K : K/K_e = \pi_1(W) \to \mathbb{R}$$

belongs to $\mathrm{Hom}(\pi_1(W), \mathbb{R})$. Since W is compact, we can identify it with a cohomology class $[\omega] \in H^1_{DR}(W)$ such that

$$\log \overline{m}_K\big([\alpha]\big) = \int_\alpha \omega \quad \text{for all } [\alpha] \in \pi_1(W).$$

Let $\pi : G \to W = G/K$ be the projection. Then $\pi^*\omega \in \Omega^1(G)$ is a closed form. Since $H^1_{DR}(G) = 0$ (G is simply connected), there exists $f : G \to \mathbb{R}$ such that $\mathrm{d}f = \pi^*\omega$. It is obvious that we can suppose $f(e) = 0$. Then the function we are looking for is $m = e^f$. In fact we have:

(a) $f(xy) = f(x) + f(y)$ for all $x \in G$, $y \in K$:
Let us consider the composition $f \circ R_y : G \to \mathbb{R}$, where R_y denotes the right traslation by the element $y \in G$. Then

$$\begin{aligned}\big(\mathrm{d}(f \circ R_y)\big)_x(v) &= (\mathrm{d}f)_{xy}(\mathrm{d}R_y)_x(v)\\ &= (\pi^*\omega)_{xy}\big((dR_y)_x(v)\big) = (R_y^*\pi^*\omega)_x(v)\\ &= (\pi^*\omega)_x(v) = (\mathrm{d}f)_x(v).\end{aligned}$$

Then $\mathrm{d}(f \circ R_y) = \mathrm{d}f$ for all $y \in K$ so, because G is connected, $f \circ R_y = f + c(y)$. Since $f(e) = 0$ we have $c(y) = f(y)$.

(b) $f_{|K} = \log \overline{m}_K$:
Let β be a path in G joining the identity e with a point $y \in K$. If we project through π we have a loop $\alpha = \pi\beta$ in W. Then,

$$\int_{\pi\beta} \omega = \log \overline{m}_K\big([\pi\beta]\big).$$

Now, by considering the isomorphism between $\pi_1(W)$ and K/K_e which sends each loop into the final point of any lifted path, it is easy to check that $\overline{m}_K([\pi\beta]) = m_K(y)$.

On the other side,

$$\int_{\pi\beta}\omega=\int_{[0,1]}(\pi\beta)^*\omega=\int_{[0,1]}\beta^*\pi^*\omega=\int_{[0,1]}\beta^*(\mathrm{d}f)=\int_{[0,1]}\mathrm{d}(\beta^*f)$$
$$=\int_{[0,1]}\mathrm{d}(f\circ\beta)=(f\beta)(1)-(f\beta)(0)=f(y).$$

This proves that $\log\overline{m}_K(y)=f(y)$ for all $y\in K$. □

Remark 1. When K is not unimodular, it may happen that the connected component K_e was unimodular, even if G/K is a compact manifold. Take for instance the linear action of $G=\mathrm{SL}(2,\mathbb{R})$ on the cylinder $C=\mathbf{R}^2-\{0\}$. Fix a real number $\lambda>0$. The quotient of C by the equivalence relation $v\equiv\lambda v$ is a torus. The isotropy of the induced action of G on T^2 is the Lie subgroup $K=K_\lambda$ of matrices

$$\begin{pmatrix}\lambda^n & t\\ 0 & \lambda^{-n}\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R}),\quad n\in\mathbb{Z},\ t\in\mathbb{R}.$$

Then $K_e=\mathbb{R}$ and the modular function of K is $m_K(n,t)=\lambda^{2n}$.

Remark 2. The proof of Theorem 2.1 in [3] implicitly assumes that the manifold W is orientable, but this may not be true. We sketch how to correct their argument.

The universal covering $\widetilde{W}=G/K_e$ is simply connected, hence orientable. Let K_2 be the group of elements $y\in K$ such that $R_y:\widetilde{W}\to\widetilde{W}$ preserves a fixed orientation. Then W is orientable if and only if $K=K_2$. If this is not the case, K_2 is a subgroup of index 2 of K such that $\Gamma_2=\Gamma\cap K_2$ is dense in K_2. Take the compact connected double covering $M_2\to M$ corresponding to $h^{-1}(\Gamma_2)\subset\pi_1(M)$. Then M_2 is an orientable manifold, and the lifted foliation $\mathcal{F}_2$ is a G-Lie foliation with holonomy Γ_2 and basic manifold $W_2=G/K_2$.

Now we must verify that $\mathcal{F}$ is unimodular if and only if $\mathcal{F}_2$ is unimodular. The basic forms for $\mathcal{F}$ are the basic forms for $\mathcal{F}_2$ which are $\mathbb{Z}_2$-invariant. Moreover to any basic form α in $\mathcal{F}_2$ we can associate the $\mathbb{Z}_2$-invariant basic form $\alpha^+=(1/2)(\alpha+L^*_{-1}\alpha)$, thus proving that the morphism $H(M/\mathcal{F})\to H(M_2/\mathcal{F}_2)$ between the basic cohomologies is injective. Hence $\mathcal{F}$ unimodular implies $\mathcal{F}_2$ unimodular. Conversely, suppose that $\mathcal{F}_2$ is unimodular. Then the morphism $H_G(G)\to H(M_2/\mathcal{F}_2)$ is injective, as shown in the same paper [3] (and previously by M. Llabrés and A. Reventós in [5], who also suppose the basic manifold to be orientable). Then $H_G(G)\to H(M/\mathcal{F})$ is injective too. Moreover the Lie group G is unimodular, hence $H^n_G(G)\neq 0$. This proves that $\mathcal{F}$ is unimodular.

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