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Probability Theory

Minima of sequences of Gaussian random variables

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Abstract

For a given sequence of real numbers $a_1, \dots, a_n$ we denote the k-th smallest one by $k\text{-}\min_{1 \leqslant i \leqslant n} a_i$. We show that there exist two absolute positive constants c and C such that for every sequence of positive real numbers $x_1, \dots, x_n$ and every $k \leqslant n$ one has

$$c \max_{1 \leqslant j \leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leqslant \mathbb{E}\, k\text{-}\min_{1 \leqslant i \leqslant n} |x_i g_i| \leqslant C \ln(k+1) \max_{1 \leqslant j \leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $g_i \in N(0,1)$, $i = 1, \dots, n$, are independent Gaussian random variables. Moreover, if $k = 1$ then the left hand side estimate does not require independence of the g_is. Similar estimates hold for $\mathbb{E}\, k\text{-}\min_{1 \leqslant i \leqslant n} |x_i g_i|^p$ as well. ***To cite this article: Y. Gordon et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Minima des suites des variables aléatoires gaussiennes. Pour une suite $a_1, \dots, a_n$ des nombres réels, on note le k-ième plus petit membre par $k\text{-}\min_{1 \leqslant i \leqslant n} a_i$. On démontre qu'il existe deux constants positives c et C telles que pour toute suite $x_1, \dots, x_n$ des nombres réels et pour tout $k \leqslant n$, on ait

$$c \max_{1 \leqslant j \leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leqslant \mathbb{E}\, k\text{-}\min_{1 \leqslant i \leqslant n} |x_i g_i| \leqslant C \ln(k+1) \max_{1 \leqslant j \leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

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Ici $g_i \in N(0,1)$, $i = 1,\dots,n$, sont des variables aléatoires Gaussiennes indépendentes. En plus, si $k = 1$, on n'a pas besoin de l'indépendence des g_i's pour obtenir l'inégalité du gauche. On démontre également les inégalités correspondantes pour $\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p$. ***Pour citer cet article : Y. Gordon et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

For a given sequence of real numbers $(a_i)_{i=1}^n$ we denote its non-decreasing rearrangement by $(k\text{-}\min_{1\leqslant i\leqslant n} a_i)_{k=1}^n$, thus $1\text{-}\min_{1\leqslant i\leqslant n} a_i = \min_{1\leqslant i\leqslant n} a_i$, $2\text{-}\min_{1\leqslant i\leqslant n} a_i$ is the next smallest, etc.

Given $A \subset \mathbb{N}$ we denote its cardinality by $|A|$. We say that $(A_j)_{j=1}^k$ is a partition of $\{1,2,\dots,n\}$ if $\emptyset \neq A_j \subset \{1,2,\dots,n\}$, $j \leqslant k$, $\bigcup_{j\leqslant k} A_j = \{1,2,\dots,n\}$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. The canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$ we denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$. By $1/t$ we mean ∞ if $t = 0$ and 0 if $t = \infty$.

In this Note we present two theorems. The first investigates the behavior of the expectation of the minimum of symmetric Gaussian random variables.

Theorem 1.1. *Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $g_i \in N(0,1)$, $i \leqslant n$, be Gaussian random variables. Then*

$$\frac{1}{1+p}\left(\frac{\pi}{2}\right)^{p/2}\left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p} \leqslant \mathbb{E} \min_{1\leqslant i\leqslant n} |x_i g_i|^p.$$

Moreover, if the g_is are independent, then

$$\mathbb{E} \min_{1\leqslant i\leqslant n} |x_i g_i|^p \leqslant \Gamma(1+p)\left(\frac{\pi}{2}\right)^{p/2}\left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p}.$$

An immediate consequence of this theorem is the following corollary:

Corollary 1.2. *Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $f_i \in N(0,1)$, $i \leqslant n$, be Gaussian random variables and $g_i \in N(0,1)$, $i \leqslant n$, be independent Gaussian random variables. Then*

$$\mathbb{E} \min_{1\leqslant i\leqslant n} |x_i g_i|^p \leqslant \Gamma(2+p)\, \mathbb{E} \min_{1\leqslant i\leqslant n} |x_i f_i|^p.$$

Remark 1. This inequality is connected to the Mallat–Zeitouni problem [2]. In fact, to prove a particular case of Conjecture 1 from [2] it is enough to prove our Corollary 1.2 for $p = 2$ and with factor 1 instead of $\Gamma(2+p)$ [3]. Thus we provide the solution of this case up to constant 6.

The next theorem deals with the moments of k-min of independent symmetric Gaussian variables.

Theorem 1.3. *Let $p > 0$. Let $2 \leqslant k \leqslant n$. Let $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$. Let $g_i \in N(0,1)$, $i \leqslant n$, be independent Gaussian random variables. Then*

$$c_p \max_{1\leqslant j\leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leqslant \left(\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p\right)^{1/p} \leqslant C(p,k) \max_{1\leqslant j\leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $c_p = \frac{1}{2\mathrm{e}}\sqrt{\frac{\pi}{2}}(1 - \frac{1}{4\sqrt{\pi}})^{1/p}$ and $C(p,k) = 4\sqrt{\pi}\max\{p, \ln(k+1)\}$.

Remark 2. Theorem 1.3 shows that we may evaluate sums of the form $\sum_{k\in I}\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n}|x_i g_i|^p$, where $I\subset\{1,2,\ldots,n\}$ is any subset of integers. Related inequalities, though in a different context, were developed initially in [1].

Theorems 1.1 and 1.3 are consequences of the following lemmas, which are of independent interest.

Lemma 1.4. *Let $0<x_1\leqslant x_2\leqslant\cdots\leqslant x_n$. Let $g_i\in N(0,1)$, $i\leqslant n$, be Gaussian random variables. Let $a=\sqrt{2/\pi}\sum_{i=1}^n 1/x_i$. Then for every $t>0$*

$$\mathbb{P}\Big\{\omega \;\Big|\; \min_{1\leqslant i\leqslant n}\big|x_i g_i(\omega)\big|\leqslant t\Big\}\leqslant at.$$

Moreover, if the g_is are independent, then for every $t>0$

$$\mathbb{P}\Big\{\omega \;\Big|\; \min_{1\leqslant i\leqslant n}\big|x_i g_i(\omega)\big|> t\Big\}\leqslant \mathrm{e}^{-at}.$$

Lemma 1.5. *Let $1\leqslant k\leqslant n$. Let $0<x_1\leqslant x_2\leqslant\cdots\leqslant x_n$. Let $g_i\in N(0,1)$, $i\leqslant n$, be independent Gaussian random variables. Let*

$$a=\frac{\mathrm{e}}{k}\sqrt{\frac{2}{\pi}}\sum_{i=1}^n\frac{1}{x_i}.$$

Then for every $0<t<1/a$ one has

$$\mathbb{P}\Big\{\omega \;\Big|\; k\text{-}\min_{1\leqslant i\leqslant n}\big|x_i g_i(\omega)\big|\leqslant t\Big\}\leqslant \frac{1}{\sqrt{2\pi k}}\,\frac{(at)^k}{1-at}. \tag{1}$$

In the rest of this Note we provide proofs of Theorems 1.1 and 1.3. Proofs of all lemmas will be shown in a forthcoming paper.

2. Proof of Theorem 1.1

Let us note that if $x_i=0$ for some i then the expectation is 0 and the theorem is trivial. Therefore, without loss of generality, we assume that $x_i>0$ for every i.

Denote $A=(\sqrt{\frac{2}{\pi}}\sum_{k=1}^n 1/x_k)^{-p}$. Then, by the first estimate in Lemma 1.4, we have $\mathbb{E}\min_{1\leqslant i\leqslant n}|x_i g_i|^p=\int_0^\infty\mathbb{P}\{\omega\mid \min_{1\leqslant i\leqslant n}|x_i g_i(\omega)|>t^{1/p}\}\,\mathrm{d}t\geqslant\int_0^A(1-t^{1/p}A^{-1/p})\,\mathrm{d}t=\frac{A}{1+p}$, which proves the first estimate.

Now assume that the g_is are independent and use the second estimate of Lemma 1.4. We obtain

$$\mathbb{E}\min_{1\leqslant i\leqslant n}|x_i g_i|^p=\int_0^\infty \mathbb{P}\Big\{\omega\;\Big|\;\min_{1\leqslant i\leqslant n}\big|x_i g_i(\omega)\big|>t^{1/p}\Big\}\,\mathrm{d}t\leqslant\int_0^\infty\exp(-t^{1/p}A^{-1/p})\,\mathrm{d}t=Ap\Gamma(p),$$

which implies the desired result. □

To prove Theorem 1.3 we need also the following combinatorial lemma:

Lemma 2.1. *Let $1\leqslant k\leqslant n$. Let $(a_i)_{i=1}^n$, be a nonincreasing sequence of positive real numbers. Then there exists a partition $(A_l)_{l\leqslant k}$ of $\{1,2,\ldots,n\}$ such that*

$$\min_{1\leqslant l\leqslant k}\sum_{i\in A_l}a_i\geqslant a:=\frac{1}{2}\min_{1\leqslant j\leqslant k}\frac{1}{k+1-j}\sum_{i=j}^n a_i.$$

Remark 3. In fact one can show that the A_ls can be taken as intervals, i.e. $A_l = \{i \mid n_{l-1} < i \leqslant n_l\}$, $l \leqslant k$, for some sequence $0 = n_0 < 1 \leqslant n_1 < n_2 < \cdots < n_k = n$.

3. Proof of Theorem 1.3

First we show the lower estimate. Since for every sequence $(a_i)_{i=1}^n$ and every $r < k$ one has $k\text{-}\min(a_i)_{i=1}^n \geqslant (k-r)\text{-}\min(a_i)_{i=r+1}^n$, it is enough to show that for every k we have

$$c_p k \left(\sum_{i=1}^n 1/x_i\right)^{-1} \leqslant \left(\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p\right)^{1/p}. \tag{2}$$

Let a be as in Lemma 1.5 and $t = (2a)^{-p}$. Then, by Lemma 1.5 and since $k \geqslant 2$, we have $\mathbb{P}\{\omega \mid k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i(\omega)|^p \geqslant t\} \geqslant 1 - \frac{1}{\sqrt{2\pi k}} \frac{(at^{1/p})^k}{1-at^{1/p}} \geqslant 1 - \frac{1}{4\sqrt{\pi}}$. Therefore (2) follows from the standard estimate $\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p \geqslant t^p \mathbb{P}\{\omega \mid k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i(\omega)| \geqslant t\}$.

Now we prove the upper bound. Let $(A_j)_{j\leqslant k}$ be the partition given by Lemma 2.1 for sequence $a_i = 1/x_i$, $i \leqslant k$. The number q, $q \geqslant 1$, will be specified later. It is easy to see that $k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p \leqslant \max_{j\leqslant k}\{\min_{i\in A_j} |x_i g_i|^p\}_{j\leqslant k}$. Therefore, using Theorem 1.1, we get

$$\left(\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p\right)^{1/p} \leqslant \left(\mathbb{E}\left(\sum_{j\leqslant k}\left(\min_{i\in A_j} |x_i g_i|^p\right)^q\right)^{1/q}\right)^{1/p} \leqslant \left(\mathbb{E} \sum_{j\leqslant k} \min_{i\in A_j} |x_i g_i|^{pq}\right)^{1/(pq)}$$
$$\leqslant \sqrt{\frac{\pi}{2}}\left(\Gamma(1+pq)\sum_{j\leqslant k}\left(\sum_{i\in A_j} 1/x_i\right)^{-pq}\right)^{1/(pq)} \leqslant \sqrt{\frac{\pi}{2}}\big(k\,\Gamma(1+pq)\big)^{1/(pq)} \max_{j\leqslant k}\left(\sum_{i\in A_j} 1/x_i\right)^{-1}.$$

Applying Lemma 2.1, we obtain $(\mathbb{E}\, k\text{-}\min_{1\leqslant i\leqslant n} |x_i g_i|^p)^{1/p} \leqslant \sqrt{2\pi}(k\Gamma(1+pq))^{1/(pq)} \max_{1\leqslant j\leqslant k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}$. To complete the proof we choose $q = \frac{\ln(k+1)}{p}$ if $p \leqslant \ln(k+1)$, $q = 1$ otherwise, and apply Stirling's formula. □

Remark 4. Finally we would like to note that our results can be extended to the case of general distributions satisfying certain conditions. Namely, fix $\alpha > 0$, $\beta > 0$ and say that a random variable ξ satisfies an (α, β)-condition if for every $t > 0$ one has $\mathbb{P}(|\xi| \leqslant t) \leqslant \alpha t$ and $\mathbb{P}(|\xi| > t) \leqslant \mathrm{e}^{-\beta t}$. Then Theorems 1.1, 1.3 and Lemmas 1.4, 1.5 hold for g_is satisfying an (α, β)-condition (even not identically distributed), with constants depending on α, β. More precisely, in the estimates of Theorem 1.1, $(\pi/2)^{p/2}$ should be substituted by α^{-p} and β^{-p} correspondingly; in Theorem 1.3, $\sqrt{\pi/2}$ should be substituted by $1/\alpha$ and, in the upper estimate, $4\sqrt{\pi}$ by $4\sqrt{2}/\beta$; in Lemma 1.5 and in the first estimate of Lemma 1.4, $\sqrt{2/\pi}$ should be substituted by α; in the second estimate of Lemma 1.4, $\sqrt{2/\pi}$ should be substituted by β.

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