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Partial Differential Equations

The Helmholtz equation with impedance in a half-plane

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Abstract

This Note gives answers to the uniqueness and existence questions for solutions of the Helmholtz equation in an half-plane with an impedance or mixed boundary condition. We deal with unbounded domains which boundaries are unbounded too. The radiation conditions are different from the ones that we found in an usual exterior problem due to the appearance of surface waves. We first compute and study the half-plane Green's function to see how the solutions behave at infinity, and second obtain integral representation for these solutions. *To cite this article: M. Duran et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).* © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

L'équation de Helmholtz avec impédance dans un demi-plan. Dans cette Note, nous démontrons un résultat d'existence et d'unicité de la solution de l'équation de Helmholtz dans un demi-plan avec une condition d'impédance. Le domaine est non borné et sa frontière également. Les conditions de radiation sont différentes des conditions usuelles pour un problème extérieur, ceci étant lié à la présence d'ondes de surface. Nous calculons la fonction de Green du demi-espace et nous étudions son comportement à l'infini. Ceci conduit à l'expression des conditions de radiation qui permettent de démontrer l'unicité. L'utilisation de la représentation intégrale donne le résultat d'existence. *Pour citer cet article : M. Duran et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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L'origine de ce problème est dans l'hydrodynamique marine. On considère un modèle très simplifié bidimensionnel d'une côte rectiligne, où la mer est représentée par un demi-espace. La hauteur d'eau dans la mer satisfait

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l'équation de Helmholtz. Une condition limite simplifiée sur la côte conduit à une condition d'impédance. Lorsque cette impédance est réelle, il apparait des ondes de surface guidées par la côte et exponentiellement décroissantes dans l'autre direction. Nous démontrons un résultat d'existence et d'unicité de la solution de l'équation de Helmholtz dans un demi-plan avec cette condition d'impédance. Les conditions de radiation sont différentes des conditions usuelles pour un problème extérieur, ceci étant lié à la présence des ondes de surface. Nous calculons la fonction de Green du demi-espace et nous étudions son comportement à l'infini. Ceci nous permet de trouver l'expression des conditions de radiation, qui permettent ensuite de démontrer l'unicité dans le cas du demi-espace. L'utilisation de la représentation intégrale donne le résultat d'existence.

1. Introduction

The origin of the problem that we want to attack is in marine hydraulics. We consider a two dimensional model of a system sea-coast where each media is represented by a half-plane. In the sea we have a time harmonic wave behavior characterized by the Helmholtz equation. On the coast, we put an impedance boundary condition saying that the normal derivative of the pressure passing through the boundary, is directly proportional to the pressure whereupon it comes.

The sea domain is not bounded. The problem is not well posed if we do not specify radiation conditions. The idea will be to obtain integral representations of solutions using a half-plane Green's function. These solutions will have to behave at infinity in the same manner that Green's function does. So firstly, we give a detailed study of the Green's function associated to the problem in order to obtain radiation conditions looking at the far field of this function. Once we specify the behavior at infinity that we want for the solutions, we are able to give answer to uniqueness and existence questions for a straight line boundary.

2. The basic model

We consider a half-plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2: y > 0\}$. Our problem is to find in this half-plane a solution for the Helmholtz equation with mixed Dirichlet–Neuman boundary data, sometimes called the impedance boundary value problem:

$$\begin{cases} \Delta u(x, y) + k^2 u(x, y) = 0 & \text{in } \mathbb{R}^2_+, \\ -\frac{\partial u}{\partial \mathbf{n}} + zu = f & \text{over } \{y = 0\}. \end{cases}$$
(1)

The normal **n** is outwardly directed, so in our problem $\frac{\partial u}{\partial \mathbf{n}}$ becomes $-\frac{\partial u}{\partial y}$. The given wave number k is positive. The impedance term z will be a positive real number that must be looked as the proportionality constant between the wave and its normal derivative. The function f must have compact support in \mathbb{R} . However, problem (1) is not well posed yet. We need to add the radiation conditions that will result from the study of the Green's function.

3. The Green's function

3.1. Determination of the square root

To determine the Fourier transform of the Green's function, we have to give an exact meaning to the complex map $\xi \mapsto \sqrt{\xi^2 - k^2}$. Briefly, we define $\sqrt{\xi^2 - k^2}$ as the product between $\sqrt{\xi - k}$ and $\sqrt{\xi + k}$. The first square root is defined using the following analytic branch of the logarithm in a region *D* composed of the whole complex



Fig. 1. $D_1 \cap D_2$, domain of the complex map $\sqrt{\xi^2 - k^2}$.

plane minus the non-negative imaginary axis. In that way, $\sqrt{\xi - k}$ is the complex square root which is analytic in the region $D_1 = D + k$.

On the other hand, $\sqrt{\xi + k}$ is a complex square root defined using the following analytic branch of the logarithm in the region D' composed of the whole complex plane minus the non-positive imaginary axis. So, $\sqrt{\xi + k}$ is analytic in the region $D_2 = D' - k$. Thus, our complex function $\sqrt{\xi^2 - k^2}$ is even and analytic in the intersection $D_1 \cap D_2$ (see Fig. 1). It has the expression

$$\sqrt{\xi^2 - k^2} = -ik \exp\left(\int_0^{\xi} \frac{\eta}{\eta^2 - k^2} \,\mathrm{d}\eta\right). \tag{2}$$

Remark 1. For real ξ , the real part of the complex map $\sqrt{\xi^2 - k^2}$ is strictly positive. Thus, the function $e^{-\sqrt{\xi^2 - k^2}y}$ is even and exponentially decreasing when $y \to +\infty$.

3.2. The expression of the Green's function

We take a source point $(x_0, y_0) \in \mathbb{R}^2_+$ and since there is no horizontal variation in the geometry of the problem, we can suppose without loss of generality that $x_0 = 0$. Our Green's function will be a solution of the boundary value problem:

$$\begin{cases} \Delta G_{y_0}(x, y) + k^2 G_{y_0}(x, y) = \delta(x, y - y_0) & \text{in } \mathbb{R}^2_+, \\ \frac{\partial G_{y_0}}{\partial y} + z G_{y_0} = 0 & \text{over } \{y = 0\}. \end{cases}$$
(3)

Taking a Fourier transform in the horizontal direction, we get the following differential equation (with initial condition) in the vertical variable *y*:

$$\begin{cases} \frac{\partial^2 \widehat{G}_{y_0}}{\partial y^2} + (k^2 - \xi^2) \widehat{G}_{y_0} = \frac{1}{\sqrt{2\pi}} \delta_{y_0} & \text{for } y > 0, \\ \frac{\partial \widehat{G}_{y_0}}{\partial y} + z \widehat{G}_{y_0} = 0 & \text{at } y = 0. \end{cases}$$

$$\tag{4}$$

The solution of (4) is called the spectral Green's function and its analytical expression is

$$\widehat{G}_{y_0}(\xi, y) = \frac{1}{\sqrt{8\pi}} \left(\frac{z + \sqrt{\xi^2 - k^2}}{z - \sqrt{\xi^2 - k^2}} \frac{e^{-\sqrt{\xi^2 - k^2}(y + y_0)}}{\sqrt{\xi^2 - k^2}} - \frac{e^{-\sqrt{\xi^2 - k^2}|y - y_0|}}{\sqrt{\xi^2 - k^2}} \right).$$
(5)

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Using the inverse Fourier transform, our spatial solution has the expression

$$G_{x_0,y_0}(x,y) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(\frac{z + \sqrt{\xi^2 - k^2}}{z - \sqrt{\xi^2 - k^2}} \frac{e^{-\sqrt{\xi^2 - k^2}(y + y_0)}}{\sqrt{\xi^2 - k^2}} - \frac{e^{-\sqrt{\xi^2 - k^2}|y - y_0|}}{\sqrt{\xi^2 - k^2}} \right) e^{-i(x - x_0)\xi} \,\mathrm{d}\xi. \tag{6}$$

Observe that now in expression (6) we have take the general case $x_0 \in \mathbb{R}$.

4. The radiation conditions

Our objective is to represent a solution of (1) as a single layer potential in terms of our boundary data and the Green's function (6). A solution like that must behave at infinite in the same way that the Green's function does. The radiation condition associated to this 'asymptotic' behavior will be sufficient to obtain the uniqueness results for the problem (1).

Throughout this section we will work with polar coordinates for the spatial variables, that is, $(x - x_0) = r \cos \varphi$ and $(y - y_0) = r \sin \varphi$. In order to find the radiation conditions for the problem (1), we want to know how the integral which expression is (6), behaves when $r \to +\infty$. Due to the absolute value, the integrand function have different behaviors depending on $y - y_0 > 0$ or $y - y_0 \leq 0$. In the first case, for any angle $0 < \varphi < \pi$ and any value of *r*, the point (x, y) belong to the half-space \mathbb{R}^2_+ . This is not true in the second case, where the far field is associated to $|x - x_0| \to +\infty$. According to that, we will find several contributions in the far field expansion.

Since $k \in \mathbb{R}$, the square root appearing in the exponentials is purely imaginary for $|\xi| < k$ and real negative for $|\xi| > k$. This suggests to split the expression (6) in two parts:

$$G_{x_0,y_0}^1(x,y) = \frac{1}{\sqrt{2\pi}} \int_{|\xi| < k} \widehat{G}_{y_0}(\xi,y) \, \mathrm{e}^{-\mathrm{i}(x-x_0)\xi} \, \mathrm{d}\xi; \quad G_{x_0,y_0}^2(x,y) := \frac{1}{\sqrt{2\pi}} \int_{|\xi| > k} \widehat{G}_{y_0}(\xi,y) \, \mathrm{e}^{-\mathrm{i}(x-x_0)\xi} \, \mathrm{d}\xi. \tag{7}$$

After some analysis techniques like the stationary phase [5] and the calculus of residues, we get the estimations

$$\begin{cases} G_{x_0,y_0}^1(x,y) = \left[\frac{z - ik\sin(\varphi)}{z + ik\sin(\varphi)} e^{2y_0 ik\sin\varphi} - 1\right] \frac{e^{i(kr + \pi/4)}}{\sqrt{8\pi kr}} + o(r^{-3/2}), & \text{when } y - y_0 > 0, \\ G_{x_0,y_0}^1(x,y) = o(r^{-1}), & \text{when } y - y_0 \leqslant 0, \\ G_{x_0,y_0}^2(x,y) = \frac{-iz}{\sqrt{z^2 + k^2}} e^{-z(y+y_0)} e^{i\sqrt{z^2 + k^2}|x - x_0|} + o(r^{-1}). \end{cases}$$
(8)

The radiation conditions allow us to choose between outgoing and incoming wave behavior. Here, we have two types of waves which lie in two different regions. This induces us to write the following general radiation condition for *r* large and $0 < \alpha < \frac{1}{2}$:

$$\begin{cases} \left| \frac{\partial u}{\partial r} - iku \right| < c r^{-(1-\alpha)} & \text{in the domain } \mathbb{R}^2_+(\alpha^+) := \{(x, y) \in \mathbb{R}^2_+; y > cr^{\alpha}\}, \\ \text{and} \\ \left| \frac{\partial u}{\partial r} - i\sqrt{k^2 + z^2} u \right| < cr^{-(1-\alpha)} & \text{in the domain } \mathbb{R}^2_+(\alpha^-) := \{(x, y) \in \mathbb{R}^2_+; y < cr^{\alpha}\}. \end{cases}$$
(9)

5. Functional spaces

Since our domains are unbounded, we need to work with weighted functional spaces. We will use powers of the classic weight functions $\rho = \sqrt{2 + r^2}$ and $\log \rho$.

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In order to adapt radiation conditions to a functional spaces framework, we start by giving a weaker version of (9) that will be sufficient for our purposes. Let ζ be a cutoff function defined in the half-plane \mathbb{R}^2_+ which is zero inside the upper unitary circle and $\zeta \equiv 1$ outside a circle of radius R > 1. We introduce the weaker radiation condition:

$$\frac{\nabla(\zeta u \,\mathrm{e}^{-\mathrm{i}kr})}{\sqrt{\rho}} \in L^2\left(\mathbb{R}^2_+(\alpha^+)\right) \quad \text{and} \quad \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial|x|} \left(\zeta u \,\mathrm{e}^{-\mathrm{i}\sqrt{z^2+k^2}|x|}\right) \in L^2\left(\mathbb{R}^2_+(\alpha^-)\right). \tag{10}$$

So we notice the space associated to problem (1) as

$$W^{1}(\mathbb{R}^{2}_{+}) = \left\{ u: \frac{u}{\rho \log \rho} \in L^{2}(\mathbb{R}^{2}_{+}), \frac{\nabla u}{\rho \log \rho} \in L^{2}(\mathbb{R}^{2}_{+}) \text{ and } u \text{ satisfies conditions (10)} \right\}.$$
(11)

Following Amrouche [2], the associated space of trace will be contained in

$$W^{1/2}(\mathbb{R}) = \left\{ u: \ \frac{u}{\sqrt{\rho}\log\rho} \in L^2(\mathbb{R}) \right\}.$$
(12)

6. The uniqueness and existence results

The following properties of the Bessel functions will be essential in the uniqueness theorem. Denote by $\{J_n(x), n \ge 0\}$, the collection of analytically extended Bessel functions of first kind and of integer order (see [1]). That is

$$J_n(x) = \begin{cases} J_n(x) & \text{if } x \ge 0, \\ (-1)^n J_n(|x|) & \text{if } x < 0. \end{cases}$$
(13)

Given k > 0, each of these functions admits the integral representation [1]

$$J_n(kx) = \frac{1}{\pi} \int_0^{\pi} \cos(kx \sin\theta - n\theta) \,\mathrm{d}\theta.$$
(14)

In other words, if we introduce the characteristic function $\chi_{(-k,k)}$ and the function $\arccos(\xi)$ with its usual determination, we have obtained the Fourier transforms

$$\left[\widehat{J_n(kx)}\right](\xi) = \mathrm{i}^{-n} \sqrt{\frac{2}{\pi}} \frac{\cos(n \arccos(-\xi/k))}{\sqrt{k^2 - \xi^2}} \chi_{(-k,k)}(\xi) \quad \text{for all } n \ge 0.$$

$$(15)$$

Theorem 6.1 (Uniqueness). The problem (1) admits a unique solution which is in $W^1(\mathbb{R}^2_+)$ (and so satisfies radiation conditions (9) or (10)).

Proof. We consider the case where f = 0. The proof is divided in several steps. The aim is to show that $u(x, 0) \equiv 0$ which also implies that $\frac{\partial u}{\partial y}(x, 0) \equiv 0$ by the boundary condition. An essential step consists of the following limit

$$\lim_{R \to +\infty} \int_{-R}^{R} u(x,0) J_n(kx) \, \mathrm{d}x = 0, \quad \text{for all } n \ge 0.$$
(16)

The proof then uses the Plancherel identity and (15). \Box

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Theorem 6.2 (Existence). Let d > 0 such that the support of f is contained in the interval [-d, d]. The function u defined by

$$u(x, y) = \int_{-d}^{d} G_{x_0,0}(x, y) f(x_0) \, \mathrm{d}x_0, \tag{17}$$

satisfies Eq. (1) and the radiation conditions (9).

Remark 2. In the paper [3], Chandler-Wilde introduced a radiation condition for a quite similar problem. However, it does not seemed to us that this covers the case when a surface wave occurs; see also the reference [4].

Remark 3. The previous result can be extended to domains which boundaries are a local perturbation of a straight line. A first step consist of extending the uniqueness and existence result to the case of a domain which is the exterior (in \mathbb{R}^2_+) of an half-circumference S^+ of radius 1, with a Dirichlet boundary data over S^+ and with the same impedance boundary condition on the lines $\{-1 < x\}$ and $\{x > 1\}$. Problems in perturbed geometries are then solved through a coupling technique using the capacity operator.

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