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Mathematical Analysis

Wiener's lemma for infinite matrices with polynomial off-diagonal decay

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Abstract

In this Note, we give a simple elementary proof to Wiener's lemma for infinite matrices with polynomial off-diagonal decay.
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Résumé

Le lemme de Wiener pour matrices infinies a decroissance polynomiale des termes non-diagonaux. Dans cette Note, nous donnons une preuve elementaire du lemme de Wiener pour les matrices infinies a decroissance polynomiale des termes non-digonaux. ***Pour citer cet article : Q. Sun, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

The classical Wiener's lemma states that *if a periodic function f has an absolutely convergent Fourier series and never vanishes, then $1/f$ has an absolutely convergent Fourier series*.

Let ℓ^p, $1 \leqslant p \leqslant \infty$, be the space of all p-summable sequences on $\mathbf{Z}^d$ equipped with usual norm $\|\cdot\|_{\ell^p}$, denote by $\mathcal{B}^2$ the space of all bounded operators on ℓ^2 equipped with usual operator norm $\|\cdot\|_{\mathcal{B}^2}$, and define $\mathcal{W} := \{(a(i-j))_{i,j\in\mathbf{Z}^d}\colon \sum_{j\in\mathbf{Z}^d} |a(j)| < \infty\}$ with a norm $\|A\|_{\mathcal{W}} := \sum_{j\in\mathbf{Z}^d} |a(j)|$ for every matrix $A = (a(i-j))_{i,j\in\mathbf{Z}^d} \in \mathcal{W}$. An equivalent formulation of the classical Wiener's lemma involving matrix algebra can be stated as follows: *$A \in \mathcal{W}$ and $A^{-1} \in \mathcal{B}^2$ imply $A^{-1} \in \mathcal{W}$*.

The classical Wiener's lemma and its various generalizations (see, for instance, [3,8,9,12–14]) are important and have numerous applications, for instance, in numerical analysis [4,17,18], wavelets and affine frames [5,14], time-

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frequency analysis [2,10–13,19], shift-invariant spaces and polynomial spline spaces [1,8,15,19], and non-uniform sampling [6,19]. Unlike the matrix algebra $\mathcal{W}$ associated with the classical Wiener's lemma, which is *commutative*, the matrix algebras in the study of spline approximation and projection [7,8], affine and Gabor frame [2,5,12,13], and non-uniform sampling [6,19] are *extremely non-commutative*. But for various purposes, we still expect that those matrix algebras have the above property that the matrix algebra $\mathcal{W}$ has.

For $p \in [1, \infty]$ and $\alpha \in \mathbf{R}$, let

$$Q_{p,\alpha} := \left\{A := \left(A(i,j)\right)_{i,j\in\mathbf{Z}^d}\colon \|A\|_{p,\alpha} < \infty\right\}, \tag{1}$$

where

$$\|A\|_{p,\alpha} := \sup_{i\in\mathbf{Z}^d} \left\|\left(A(i,j)\left(1+|i-j|\right)^{\alpha}\right)_{j\in\mathbf{Z}^d}\right\|_{\ell^p} + \sup_{j\in\mathbf{Z}^d} \left\|\left(A(i,j)\left(1+|i-j|\right)^{\alpha}\right)_{i\in\mathbf{Z}^d}\right\|_{\ell^p}. \tag{2}$$

For $p = \infty$, we see that $A = (A(i,j))_{i,j\in\mathbf{Z}^d} \in Q_{\infty,\alpha}$ if and only if $|A(i,j)| \leqslant \|A\|_{\infty,\alpha}(1+|i-j|)^{-\alpha}$ for all $i, j \in \mathbf{Z}^d$. Because of the above interpretation of matrices in $Q_{p,\alpha}$ for $p = \infty$, we call matrices in $Q_{p,\alpha}$ to have *polynomial off-diagonal decay*.

For the matrix algebra $Q_{p,\alpha}$ with $p = \infty$ and $\alpha > d$, Jaffard use a rather delicate bootstrap argument to prove that $A \in Q_{\infty,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{\infty,\alpha}$ [14]. For the matrix algebra $Q_{p,\alpha}$ with $p = 1$ and $\alpha > 0$, Barnes use the Banach algebra technique to show that $A \in Q_{1,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{1,\alpha}$ (see [3] for $\alpha \in (0,1]$ and [13] for any $\alpha > 0$). In this Note, we study the matrix algebra $Q_{p,\alpha}$ with $1 \leqslant p \leqslant \infty$ and $\alpha > d(1-1/p)$ and give a simple elementary proof to the following Wiener's lemma.

Theorem 1.1. *Let* $1 \leqslant p \leqslant \infty$ *and* $\alpha > d(1-1/p)$. *Then* $A \in Q_{p,\alpha}$ *and* $A^{-1} \in \mathcal{B}^2$ *imply* $A^{-1} \in Q_{p,\alpha}$.

More general formulation of the above Wiener's lemma and its applications to frames and sampling will be discussed in the subsequent paper [19].

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *Let* $1 \leqslant p \leqslant \infty$ *and* $\alpha > d(1-1/p)$. *Then there exist positive constants* C_1 *and* C_2 *such that*

$$\|A^n\|_{p,\alpha} \leqslant C_1\left(C_2\frac{\|A\|_{p,\alpha}}{\|A\|_{\mathcal{B}^2}}\right)^{\frac{2-\theta}{1-\theta}n^{\log_2(2-\theta)}}\left(\|A\|_{\mathcal{B}^2}\right)^n \tag{3}$$

holds for all $A \in Q_{p,\alpha}$ *and* $n \geqslant 1$, *where* $\theta = 1 - \frac{d}{2\alpha-2d(1/2-1/p)}$.

Proof. By Hölder inequality,

$$\|A\|_{1,0} \leqslant C\|A\|_{p,\alpha} \quad \text{for all } A \in Q_{p,\alpha}. \tag{4}$$

Here and hereafter, the letter C denotes an absolute constant which could be different at different occurrence.

By the definition of the operator norm $\|\cdot\|_{\mathcal{B}^2}$,

$$\|A\|_{2,0} \leqslant \|A\|_{\mathcal{B}^2} \leqslant \|A\|_{1,0} \quad \text{for all } A \in Q_{1,0}. \tag{5}$$

For any $A = (A(i,j))_{i,j\in\mathbf{Z}^d}$ and $B = (B(i,j))_{i,j\in\mathbf{Z}^d}$ in $Q_{p,\alpha}$,

$$\|AB\|_{p,\alpha} \leqslant 2^{\alpha}\|A\|_{p,\alpha}\|B\|_{1,0} + 2^{\alpha}\|A\|_{1,0}\|B\|_{p,\alpha}, \tag{6}$$

by Hölder inequality and the following estimate:

$$\left|(AB)(i,j)\right|\left(1+|i-j|\right)^{\alpha} \leqslant 2^{\alpha}\sum_{k\in\mathbf{Z}^d}\left|A(i,k)\right|\left(1+|i-k|\right)^{\alpha}\left|B(k,j)\right| + 2^{\alpha}\sum_{k\in\mathbf{Z}^d}\left|A(i,k)\right|\left|B(k,j)\right|\left(1+|k-j|\right)^{\alpha}.$$

Let $\theta_1=(\alpha-d(1/2-1/p))^{-1}$ and $\tau=(\|A\|_{p,\alpha})^{\theta_1}(\|A\|_{\mathcal{B}^2})^{-\theta_1}$. Then

$$\sum_{k\in\mathbf{Z}^d}\left|A(i,k)\right| \leqslant \sum_{|i-k|\leqslant\tau}\left|A(i,k)\right| + \sum_{|i-k|\geqslant\tau}\left|A(i,k)\right| \leqslant C\tau^{d/2}\|A\|_{2,0}+C\tau^{-\alpha+d(1-1/p)}\|A\|_{p,\alpha}$$
$$\leqslant C\tau^{d/2}\|A\|_{\mathcal{B}^2}+C\tau^{-\alpha+d(1-1/p)}\|A\|_{p,\alpha} = 2C\left(\|A\|_{\mathcal{B}^2}\right)^{1-d\theta_1/2}\left(\|A\|_{p,\alpha}\right)^{d\theta_1/2}$$

by (4) and (5), which yields

$$\|A\|_{1,0}\leqslant C\left(\|A\|_{\mathcal{B}^2}\right)^{1-d\theta_1/2}\left(\|A\|_{p,\alpha}\right)^{d\theta_1/2} \quad \text{for all } A\in Q_{p,\alpha}. \tag{7}$$

Combining (6) and (7) leads to the following compensated compactness estimate:

$$\|A^2\|_{p,\alpha}\leqslant C\|A\|_{p,\alpha}^{2-\theta}\|A\|_{\mathcal{B}^2}^{\theta} \quad \text{for all } A\in Q_{p,\alpha}. \tag{8}$$

Applying (4), (6), and (8), and using $\|A^n\|_{\mathcal{B}^2}\leqslant\|A\|_{\mathcal{B}^2}^n$ for $n\geqslant 1$, we obtain the following for any $n\geqslant 1$:

$$\|A^{2n}\|_{p,\alpha}\leqslant D\left(\|A^n\|_{p,\alpha}\right)^{2-\theta}\left(\|A\|_{\mathcal{B}^2}\right)^{n\theta},$$

and

$$\|A^{2n+1}\|_{p,\alpha}\leqslant D\|A\|_{p,\alpha}\left(\|A^n\|_{p,\alpha}\right)^{2-\theta}\left(\|A\|_{\mathcal{B}^2}\right)^{n\theta},$$

where $D\geqslant 1$ is a positive constant independent of $A\in Q_{p,\alpha}$ and $n\geqslant 1$. Thus the sequence $\{b_n\}$, to be defined by $b_n=D^{1/(1-\theta)}\|A^n\|_{p,\alpha}(\|A\|_{\mathcal{B}^2})^{-n}$, $n\geqslant 1$, satisfies

$$b_{2n}\leqslant b_n^{2-\theta} \quad \text{and} \quad b_{2n+1}\leqslant b_1 b_n^{2-\theta} \quad \text{for all } n\geqslant 1.$$

By induction, we have the following upper bound estimate to the sequence $\{b_n\}$:

$$b_n\leqslant b_1^{\sum_{i=0}^{l}\epsilon_i(2-\theta)^i}\leqslant b_1^{\frac{2-\theta}{1-\theta}n^{\log_2(2-\theta)}}$$

for $n=\sum_{i=0}^{l}\epsilon_i 2^i$, where $\epsilon_i\in\{0,1\}$, $0\leqslant i\leqslant l$. Therefore (3) follows. □

Remark 1. For the special case that $p=1$, $\alpha=0$, and $A=(q(j-j'))_{j,j'\in\mathbf{Z}}$ with $\sum_{j\in\mathbf{Z}}q(j)\,\mathrm{e}^{-ij\xi}$ being reciprocal of a trigonometric polynomial Q, Newman proved the following better estimate than the one in (3) for the $Q_{1,0}$ norm of A^n: $\|A^n\|_{1,0}\leqslant Cn^2\|A\|_{\mathcal{B}^2}^n$ for all $n\geqslant 1$, where C is a positive constant depending on the degree of the polynomial Q. That estimate is crucial for Newman's elementary proof of the classical Wiener's lemma [16].

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. For any $A=(A(i,j))_{i,j\in\mathbf{Z}^d}\in Q_{p,\alpha}$, we define its transpose A^* by $A^*:=(\overline{A(j,i)})_{i,j\in\mathbf{Z}^d}$. Then $A^*A\in Q_{p,\alpha}$ by (4), (6), and the fact that $\|A^*\|_{p,\alpha}=\|A\|_{p,\alpha}$. This, together with the fact that A^*A is a positive operator on ℓ^2 by the assumption on the matrix A, implies that

$$A^*A=\|A^*A\|_{\mathcal{B}^2}(I-B) \tag{9}$$

for some $B\in\mathcal{B}^2$ with

$$\|B\|_{\mathcal{B}^2}<1 \quad \text{and} \quad \|B\|_{p,\alpha}<\infty, \tag{10}$$

where I is the identity operator on ℓ^2. By (10) and Lemma 2.1, we obtain

$$\left\|(I-B)^{-1}\right\|_{p,\alpha} \leqslant \sum_{n=0}^{\infty} \|B^n\|_{p,\alpha} \leqslant \sum_{n=0}^{\infty} C_1 \left(C_2 \frac{\|B\|_{p,\alpha}}{\|B\|_{\mathcal{B}^2}}\right)^{\frac{2-\theta}{1-\theta} n^{\log_2(2-\theta)}} \left(\|B\|_{\mathcal{B}^2}\right)^n < \infty. \tag{11}$$

The conclusion $A^{-1} \in Q_{p,\alpha}$ then follows from (4), (6), (9), (11), and the fact that $A^{-1} = (A^*A)^{-1}A^*$. $\square$

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