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Algebra

The structure of certain rigid tensor categories

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Abstract

We consider rigid tensor categories over a field of characteristic zero in which some exterior power of each object is zero. ***To cite this article: P. O'Sullivan, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

La structure de certaines catégories tensorielles rigides. Nous considérons des catégories tensorielles rigides sur un corps de caractéristique nulle dans lesquelles une puissance extérieure convenable de chaque objet est nulle. ***Pour citer cet article : P. O'Sullivan, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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Soit k un corps de caractéristique 0. Disons qu'une catégorie k-tensorielle rigide $\mathcal{A}$ est *positive* si pour chaque objet M de $\mathcal{A}$ on a $\bigwedge^r M = 0$ pour r convenable. Chaque catégorie k-tensorielle rigide $\mathcal{A}$ telle que $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$ a un idéal tensoriel maximal unique $\mathcal{R}(\mathcal{A})$. Si $\mathcal{A}$ est de plus positive, alors $\mathcal{A}/\mathcal{R}(\mathcal{A})$ est positive et semi-simple. Le résultat principal est le Théorème 0.1 ci-dessous. Une démonstation totalement différente de ce résultat a été donnée par André et Kahn dans [1].

Théorème 0.1 (cf. [1], 16.1.1 et 13.7.1). *Soit $\mathcal{A}$ une catégorie k-tensorielle positive avec $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$. Alors la projection $\mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$ admet un quasi-inverse à droite. Si T est un tel quasi-inverse à droite et si $\mathcal{D}$ est une catégorie k-tensorielle positive semi-simple avec $\mathrm{End}_{\mathcal{D}}(\mathbf{1}) = k$, alors tout foncteur k-tensoriel $\mathcal{D} \to \mathcal{A}$ se factorise, à isomorphisme tensoriel près, à travers T.*

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Nous prouvons en fait le Lemme 0.2 ci-dessous. Le Théorème 0.1 dit de plus que deux quasi-inverses à droite quelconques T_1 et T_2 de la projection coïncident à isomorphisme tensoriel près, mais ceci se déduit du Lemme 0.2 en prenant $\mathcal{D} = \mathcal{A}/\mathcal{R}(\mathcal{A}) \otimes_k \mathcal{A}/\mathcal{R}(\mathcal{A})$ et pour $\mathcal{D} \to \mathcal{A}$ le foncteur défini par T_1 et T_2.

Lemme 0.2. *Si $\mathcal{A}$ et $\mathcal{D}$ sont comme dans le Théorème* 0.1, *alors tout foncteur k-tensoriel $\mathcal{D} \to \mathcal{A}$ se factorise, à isomorphisme tensoriel près, à travers un quasi-inverse à droite à la projection $\mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$.*

Pour prouver le Lemme 0.2 on peut supposer, par les deux lemmes suivants, que $\mathcal{A} = \mathrm{Mod}(A)$ et que $\mathcal{D} \to \mathcal{A}$ est $A \otimes -$ pour A une algèbre (commutative) dans la catégorie $\mathrm{Ind}(\mathcal{D})$ des ind-objets de $\mathcal{D}$, où $\mathrm{Mod}(A)$ est l'enveloppe pseudo-abélienne de la catégorie de A-modules libres $A \otimes M$ avec $M \in \mathrm{Ob}\,\mathcal{D}$.

Lemme 0.3. *Soit $(M_i)_{i \in I}$ une famille d'objets dans une catégorie k-tensorielle positive $\mathcal{A}$, avec* $\dim M_i = n_i \in \mathbf{N}$. *Notons V_i la représentation standard du facteur* $\mathrm{GL}(n_i)$ *de* $\prod_{i \in I} \mathrm{GL}(n_i)$. *Alors il existe un foncteur k-tensoriel $H : \mathrm{Rep}_k(\prod_{i \in I} \mathrm{GL}(n_i)) \to \mathcal{A}$ avec $HV_i = M_i$ pour $i \in I$.*

Lemme 0.4. *Soient $H : \mathcal{C} \to \mathcal{A}$ un foncteur k-tensoriel. Supposons que $\mathcal{C}$ soit rigide et que* $\mathrm{Hom}_{\mathcal{A}}(H-, \mathbf{1})$ *sur $\mathcal{C}$ soit ind-representable. Alors H est, à isomorphisme tensoriel près, le composé d'un foncteur k-tensoriel $A \otimes - : \mathcal{C} \to \mathrm{Mod}(A)$ avec un foncteur k-tensoriel pleinement fidèle $\mathrm{Mod}(A) \to \mathcal{A}$.*

Pour prouver le Lemme 0.2 lorsque $\mathcal{D} \to \mathcal{A}$ est $A \otimes - : \mathcal{D} \to \mathrm{Mod}(A)$, soit $\bar{A}$ un quotient simple de A. Par le Lemme 0.5 ci-dessous on peut identifier $\mathcal{A}/\mathcal{R}(\mathcal{A})$ et $\mathrm{Mod}(\bar{A})$ de sorte que $\mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$ soit défini par la projection $A \to \bar{A}$. Le Lemme 0.2 résulte donc du Lemme 0.6.

Lemme 0.5. *Soient $\mathcal{D}$ une catégorie k-tensorielle positive semi-simple avec* $\mathrm{End}_{\mathcal{D}}\,\mathbf{1} = k$ *et A une algèbre dans* $\mathrm{Ind}(\mathcal{D})$ *avec* $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(\mathbf{1}, A) = k$. *Alors* $\mathrm{Mod}(A)$ *est semi-simple si et seulement si A est simple.*

Lemme 0.6. *Si $\mathcal{D}$ et A sont comme dans le Lemme* 0.5, *alors la projection de A sur un quotient simple admet un inverse à droite dans la catégorie des algèbres dans* $\mathrm{Ind}(\mathcal{D})$.

Soit $\mathcal{A}$ une catégorie k-tensorielle positive. Par les Lemmes 0.3 et 0.4, il existe un couple (G, X), avec G un k-groupe pro-réductif et X un G-schéma affine, tel que $\mathcal{A}$ soit $\otimes$-équivalente à la catégorie des fibrés vectoriels G-équivariants sur X. Si $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$ et si k est algébriquement clos, on peut déduire du Théorème 0.1 qu'il existe un tel (G, X) unique à isomorphisme près tel que X ait un k-point fixé par G.

1. Introduction

Fix a field k of characteristic 0. By a k-tensor category we mean a k-linear, pseudo-Abelian, symmetric monoidal category. The exterior power $\bigwedge^r M$ of an object M in a k-tensor category $\mathcal{A}$ is defined as the image of the anti-symmetriser in $\mathrm{End}_{\mathcal{A}}(M^{\otimes r})$. We say that M is *positive* if M has a dual $M^\vee$ and $\bigwedge^r M = 0$ for some r, and that $\mathcal{A}$ is positive if it is essentially small with every object positive. A k-tensor category is said to be *rigid* if every object has a dual. Let $\mathcal{A}$ be a rigid k-tensor category with $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$. Then $\mathcal{A}$ has a unique maximal tensor ideal $\mathcal{R}(\mathcal{A})$, consisting of the $f : M \to N$ such that $\mathrm{tr}(fg) = 0$ for each $g : N \to M$. We have $\mathcal{R}(\mathcal{A}) = 0$ if and only if each non-zero $M \to \mathbf{1}$ in $\mathcal{A}$ is a retraction.

A category will be called *semisimple* if it is Abelian with every object projective. By [3], Théorème 7.1, a k-tensor category is semisimple positive with $\mathrm{End}(\mathbf{1}) = k$ if and only if it is semisimple Tannakian. Theorem 1.1 is proved in Section 6, and in Section 7 it is indicated how structure theorems can be deduced from it. A totally different proof of Theorem 1.1 has been given by André and Kahn in [1].

Theorem 1.1 (cf. [1], 16.1.1 and 13.7.1). *Let $\mathcal{A}$ be a positive k-tensor category with $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$. Then $\mathcal{A}/\mathcal{R}(\mathcal{A})$ is semisimple and positive, and the projection $\mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$ has a right quasi-inverse. If T is such a right quasi-inverse and $\mathcal{D}$ is a semisimple positive k-tensor category with $\mathrm{End}_{\mathcal{D}}(\mathbf{1}) = k$, then any k-tensor functor $\mathcal{D} \to \mathcal{A}$ factors up to tensor isomorphism through T.*

2. Algebras and modules in a tensor category

Let $\mathcal{A}$ be a k-tensor category. If A is an algebra (always understood to be commutative) in $\mathcal{A}$, we denote by $\mathrm{MOD}(A)$ the category of A-modules. When $\mathcal{A}$ is Abelian and its tensor product preserves colimits, $\mathrm{MOD}(A)$ is an Abelian k-tensor category and the forgetful functor $\mathrm{MOD}(A) \to \mathcal{A}$ creates limits and colimits. We say that the algebra A is *simple* if it is simple as an object of $\mathrm{MOD}(A)$.

Suppose that $\mathcal{A}$ is the category $\mathrm{Ind}(\mathcal{C})$ of ind-objects ([2], I 8) of a k-tensor category $\mathcal{C}$. Then we denote by $\mathrm{Mod}(A)$ the pseudo-Abelian hull of the full subcategory of $\mathrm{MOD}(A)$ of free A-modules $A \otimes M$ on objects M in $\mathcal{C}$. It is a k-tensor category. We have a k-tensor functor $A \otimes - : \mathcal{C} \to \mathrm{Mod}(A)$, and also $B \otimes_A - : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ defined by a given $A \to B$. If $\mathcal{C}$ is positive then $\mathrm{Mod}(A)$ is positive.

Let G be an affine k-group. We denote by $\mathrm{Rep}_k(G)$ the k-tensor category of finite-dimensional representations of G, and by $\mathrm{REP}_k(G)$ the k-tensor category of all representations. We have $\mathrm{REP}_k(G) = \mathrm{Ind}(\mathrm{Rep}_k(G))$. An algebra A in $\mathrm{REP}_k(G)$ is the same as a G-algebra, and we write $\mathrm{MOD}(G, A)$ for $\mathrm{MOD}(A)$ and $\mathrm{Mod}(G, A)$ for $\mathrm{Mod}(A)$. The k-group G is proreductive if and only if $\mathrm{Rep}_k(G)$ (or equivalently $\mathrm{REP}_k(G)$) is semisimple.

3. Positive objects and representations of the general linear group

Lemma 3.1. *The dimension $\dim M = \mathrm{tr}\, 1_M \in \mathrm{End}_{\mathcal{A}}(\mathbf{1})$ of a positive object M in a k-tensor category $\mathcal{A}$ can be written as $\sum_{j=1}^{s} n_j e_j$, where $n_j \in \mathbf{N}$ and the e_j are mutually orthogonal idempotents of $\mathrm{End}_{\mathcal{A}}(\mathbf{1})$ with $\sum_{j=1}^{s} e_j = 1$. If $e_j \neq 0$ for all j, then $\max_{1 \leqslant j \leqslant s} n_j$ is the least $m \in \mathbf{N}$ such that $\bigwedge^{m+1} M = 0$.*

Proof. (For the case where $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$ see also [1], 9.1.7.) We may suppose $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) \neq 0$. Write $\alpha_r \in \mathrm{End}_{\mathcal{A}}(M^{\otimes r})$ for the antisymmetriser, and $u : \mathbf{1} \to M \otimes M^{\vee}$ and $c : M \otimes M^{\vee} \to \mathbf{1}$ for the canonical morphisms. If $\dim M = d$, the "contraction" $(M^{\otimes r} \otimes c) \circ (\alpha_{r+1} \otimes M^{\vee}) \circ (M^{\otimes r} \otimes u)$ of α_{r+1} is $(d-r)/(r+1)\alpha_r$. Suppose that $\alpha_{m+1} = 0$ and $\alpha_m \neq 0$. Then inductively we have $\binom{d}{m+1} = \mathrm{tr}\, \alpha_{m+1} = 0$, whence the first statement, with $n_j \leqslant m$ when $e_j \neq 0$. Also $(d-m)\alpha_m = 0$, whence the second statement. □

Lemma 3.2. *Let $(M_i)_{i \in I}$ be a family of positive objects in a k-tensor category $\mathcal{A}$, with $\dim M_i = m_i \in \mathbf{N}$. Denote by V_i the standard representation of the factor $\mathrm{GL}(m_i)$ of $\prod_{i \in I} \mathrm{GL}(m_i)$. Then there is a k-tensor functor $H : \mathrm{Rep}_k(\prod_{i \in I} \mathrm{GL}(m_i)) \to \mathcal{A}$ with $HV_i = M_i$ for $i \in I$.*

Proof. For simplicity we give the proof in the case where I has one element, and omit the indices i. The general case is similar, with the free rigid k-tensor category on a family $(N)_{i \in I}$ replacing $\mathcal{F}$ below. Let $\mathfrak{S}_r$ be the symmetric group of degree r, and write $a_r \in k[\mathfrak{S}_r]$ for the antisymmetriser and $\sigma_L^r : k[\mathfrak{S}_r] \to \mathrm{End}(L^{\otimes r})$ for the canonical homomorphism associated to an object L. Let $\mathcal{F}$ be the free rigid k-tensor category on one object N (see [4], 1.26). We have $\mathrm{End}_{\mathcal{F}}(\mathbf{1}) = k[t]$ with $t = \dim N$, the σ_N^r induce isomorphisms $k[t][\mathfrak{S}_r] \simeq \mathrm{End}_{\mathcal{F}}(N^{\otimes r})$, and $\mathrm{Hom}_{\mathcal{F}}(N^{\otimes r}, N^{\otimes s}) = 0$ for $r \neq s$. If $\mathcal{J}$ is the tensor ideal of $\mathcal{F}$ generated by $t - m$ and $\sigma_N^{m+1} a_{m+1}$, let $\overline{\mathcal{F}}$ be the pseudo-Abelian hull of $\mathcal{F}/\mathcal{J}$, and $\overline{N}$ the image of N in $\overline{\mathcal{F}}$. Since $\dim V = \dim M = m$ and $\sigma_V^{m+1} a_{m+1} = 0$ and (by Lemma 3.1) $\sigma_M^{m+1} a_{m+1} = 0$, there are by the universal property of $\mathcal{F}$ k-tensor functors $R : \overline{\mathcal{F}} \to \mathrm{Rep}_k(\mathrm{GL}(m))$ and $S : \overline{\mathcal{F}} \to \mathcal{A}$ with $R\overline{N} = V$ and $S\overline{N} = M$. Now σ_V^r is surjective, with kernel 0 for $r \leqslant m$ and generated by $a_{m+1} \in k[\mathfrak{S}_{m+1}] \subset k[\mathfrak{S}_r]$ for $r > m$ (e.g. [5], Theorem 6.3). Also $\sigma_{\overline{N}}^r$ is surjective with kernel containing a_{m+1} for $r > m$. Thus R induces an isomorphism on the $\mathrm{End}_{\overline{\mathcal{F}}}(\overline{N}^{\otimes r})$. We have $\mathrm{Hom}_{\mathrm{GL}(m)}(V^{\otimes r}, V^{\otimes s}) = 0$ for $r \neq s$, so R is an equivalence by rigidity of $\overline{\mathcal{F}}$. Now take $H = S \circ R'$ with R' quasi-inverse to R. □

Lemma 3.3. *Let $H : \mathcal{C} \to \mathcal{A}$ be a k-tensor functor. Suppose that $\mathcal{C}$ is rigid and that* $\mathrm{Hom}_{\mathcal{A}}(H-, \mathbf{1})$ *on $\mathcal{C}$ is ind-representable* ([2], I 8.2.2). *Then H is tensor isomorphic to the composite of a k-tensor functor $A \otimes - : \mathcal{C} \to$* $\mathrm{Mod}(A)$ *with a fully faithful k-tensor functor* $\mathrm{Mod}(A) \to \mathcal{A}$.

Proof. Write $\widetilde{H} = \mathrm{Ind}(H)$ and let $f : \widetilde{H}A \to \mathbf{1}$ define an isomorphism $\varphi : \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(-, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}(\widetilde{H}-, \mathbf{1})$. There is a unique algebra structure on A such that f is a morphism of algebras. Then H is tensor isomorphic to $F \circ (A \otimes -)$ with $F = \mathbf{1} \otimes_{\widetilde{H}A} \widetilde{H}- : \mathrm{Mod}(A) \to \mathcal{A}$ defined by f. The factorisation $\mathrm{Hom}_A(A \otimes M, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(M, A) \xrightarrow{\varphi_M} \mathrm{Hom}_{\mathcal{A}}(HM, \mathbf{1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(F(A \otimes M), FA)$ of $F_{A \otimes M, A}$ for $M \in \mathcal{C}$, together with the rigidity of $\mathrm{Mod}(A)$, shows that F is fully faithful. □

Lemma 3.4. *A k-tensor category $\mathcal{A}$ is positive if and only if there exists a proreductive k-group G and a G-algebra A such that $\mathcal{A}$ is k-tensor equivalent to* $\mathrm{Mod}(G, A)$.

Proof. Suppose that $\mathcal{A}$ is positive. If M is an object of $\mathcal{A}$, and if n_j and e_j are as in Lemma 3.1, then since the image N_j of $e_j : \mathbf{1} \to \mathbf{1}$ has dimension e_j, the object $M \oplus \bigoplus_j N_j^{m-n_j}$ has dimension $m \in \mathbf{N}$ for $m \geqslant \max_j n_j$. Hence by Lemma 3.2 there is a $G = \prod_{i \in I} \mathrm{GL}(m_i)$ and a k-tensor functor $H : \mathrm{Rep}_k(G) \to \mathcal{A}$ such that every object of $\mathcal{A}$ is a direct summand of one in the image of H. Since $\mathrm{Rep}_k(G)$ is semisimple, $\mathrm{Hom}_{\mathcal{A}}(H-, \mathbf{1})$ is exact and hence ind-representable ([2], I 8.3.3). Thus Lemma 3.3 gives a k-tensor equivalence $\mathrm{Mod}(G, A) \to \mathcal{A}$. The converse is immediate. □

4. Simple algebras and semisimplicity

Lemma 4.1 (see [1] 8.2.4). *Let $\mathcal{A}$ be a positive k-tensor category with* $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$. *Then $\mathcal{A}/\mathcal{R}(\mathcal{A})$ is semisimple and positive with finite-dimensional hom k-spaces. If $\mathcal{A}$ is semisimple then* $\mathcal{R}(\mathcal{A}) = 0$.

Lemma 4.2. *Let $\mathcal{D}$ be a semisimple positive k-tensor category with* $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$ *and let A be an algebra in* $\mathrm{Ind}(\mathcal{D})$ *with* $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(\mathbf{1}, A) = k$. *Then* $\mathrm{Mod}(A)$ *is semisimple if and only if A is simple. When this is so* $\mathrm{MOD}(A)$ *is semisimple, and* $\mathrm{MOD}(A) = \mathrm{Ind}(\mathrm{Mod}(A))$.

Proof. The isomorphisms $\mathrm{Hom}_A(A \otimes N, -) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{D})}(N, -)$ show that $\mathrm{Hom}_A(L, -)$ is exact and preserves filtered colimits when $L \in \mathrm{Mod}(A)$. Now A is simple if and only if each non-zero $M \to A$ in $\mathrm{MOD}(A)$ is an epimorphism, and by Lemma 4.1 $\mathrm{Mod}(A)$ is semisimple if and only if each non-zero $M \to A$ in $\mathrm{Mod}(A)$ is a retraction. Since $\mathrm{Hom}_A(A, -)$ is exact, the first statement follows. Suppose now that $\mathrm{Mod}(A)$ is semisimple. Any A-module L is the cokernel of an A-morphism $l : A \otimes K' \to A \otimes K$, with for example $K = L$. Writing K and K' as filtered colimits of objects in $\mathcal{D}$ and appropriately reindexing, we may express l as a filtered colimit of A-morphisms $l_\lambda : A \otimes K'_\lambda \to A \otimes K_\lambda$ with $K_\lambda, K'_\lambda \in \mathcal{D}$. Then L is the filtered colimit of the cokernels of the l_λ, which by semisimplicity lie in $\mathrm{Mod}(A)$. Thus $\mathrm{MOD}(A) = \mathrm{Ind}(\mathrm{Mod}(A))$. The objects of $\mathrm{Mod}(A)$ are of finite length by Lemma 4.1, so each object of $\mathrm{MOD}(A)$ is a coproduct of simple objects of $\mathrm{Mod}(A)$, whence $\mathrm{MOD}(A)$ is semisimple. □

5. Algebras with action of a proreductive group

Lemma 5.1 (Magid [7], Theorem 4.5). *Let G be an affine k-group of finite type and let A be a G-algebra with $A^G = k$. Then A is a simple G-algebra if and only if* $\mathrm{Spec}(A)$ *is homogeneous under G.*

Lemma 5.2. *Let G be a reductive k-group, let A be a finitely generated G-algebra with $A^G = k$, and let $J \neq A$ be a G-ideal of A. Then the canonical homomorphism $A \to \lim_n A/J^n$, where the limit is taken in* $\mathrm{REP}_k(G)$, *is an isomorphism.*

Proof. It suffices to show that $\mathrm{Hom}_G(V, A) \to \lim_n \mathrm{Hom}_G(V, A/J^n)$ is bijective for $V \in \mathrm{Rep}_k(G)$. The surjectivity is clear since $\mathrm{Hom}_G(V, A)$ is finite-dimensional over $k = A^G$ (e.g. [8], Theorem 3.25) and $\mathrm{Hom}_G(V, -)$ is exact. To prove the injectivity, we may by extending the scalars assume k algebraically closed. It suffices to check that then $\bigcap_n J^n = 0$, or equivalently (e.g. [9], Chapter IV, Theorem 12′) that if $\mathfrak{p}$ is an associated prime of $0 \subset A$ then $J + \mathfrak{p} \neq A$. Since k is algebraically closed, a $G(k)$-subspace of a representation of G is a G-subspace, as can be seen by reducing to the finite-dimensional case. Thus $\mathfrak{p}_0 = \bigcap_{g \in G(k)} g\mathfrak{p}$ is a G-ideal of A. It follows that $A \to A/J \times A/\mathfrak{p}_0$ is not surjective since $\dim_k (A/J \times A/\mathfrak{p}_0)^G > 1$, so $J + \mathfrak{p}_0 \neq A$. Since each $g\mathfrak{p}$ lies in the finite set of associated primes of 0, some $J + g\mathfrak{p} \neq A$, so $J + \mathfrak{p} = g^{-1}(J + g\mathfrak{p}) \neq A$. □

Lemma 5.3 (cf. [6], Corollaire 2). *Let G be a proreductive k-group and A be a G-algebra with $A^G = k$. Then A has a unique simple G-quotient $\bar{A}$. If D is a simple G-subalgebra of A, then the projection $A \to \bar{A}$ has a right inverse in the category of G-algebras over D.*

Proof. By Zorn's Lemma A has a maximal G-ideal J. If $J' \neq A$ is a G-ideal of A, then $A \to A/J \times A/J'$ is not surjective since $\dim_k (A/J \times A/J')^G > 1$, so $J + J' \neq A$ and $J' \subset J$. Thus $\bar{A} = A/J$ exists and is unique. The second statement is proved in (1), (2) and (3) below. We note that if G is of finite type then by Lemma 5.1 $\bar{A}$ and D are finitely generated k-algebras and $\bar{A}$ is smooth over D.

(1) Suppose that G is of finite type and that $J^2 = 0$. Write E for the set of k-algebra homomorphisms $\bar{A} \to A$ over D right inverse to $A \to \bar{A}$, and let $V \subset \bar{A}$ be a finite-dimensional G-subspace which contains k and generates $\bar{A}$. We may regard E as a subset of $\mathrm{Hom}_k(V, A) = V^\vee \otimes_k A$. Now $E \neq \emptyset$ by smoothness of $\bar{A}$ over D, and if $e \in E$ then $E - e$ is the k-space of derivations of $\bar{A}$ over D with values in J. Thus the k-subspace $\widetilde{E}$ of $V^\vee \otimes_k A$ generated by E is a G-subspace, and the evaluation $V^\vee \otimes_k A \to A$ at $1 \in V$ defines a surjective G-homomorphism $\widetilde{E} \to k \subset A$ with fibre E above $1 \in k$. Since G is reductive, the set $\widetilde{E}^G \cap E$ of homomorphisms of G-algebras $\bar{A} \to A$ over D right inverse to $A \to \bar{A}$ is non-empty.

(2) Suppose that G is of finite type. Replacing A with its G-subalgebra generated by D and a lifting to A of a finite set of generators of $\bar{A}$, we may assume that A is finitely generated. Then $A = \lim_n A/J^n$ in $\mathrm{REP}(G)$ by Lemma 5.2. Thus it is enough to show that a morphism $\bar{A} \to A/J^n$ of G-algebras over D can be lifted to $\bar{A} \to A/J^{n+1}$. In fact the pullback of $A/J^{n+1} \to A/J^n$ along $\bar{A} \to A/J^n$ has kernel of square 0, and so has a right inverse over D by (1).

(3) Consider the general case. By Zorn's Lemma, A has a maximal simple G-subalgebra C containing D. It suffices to show that $C \to \bar{A}$ is an isomorphism. To do this we show that $C^H \to \bar{A}^H$ is an isomorphism for each normal k-subgroup H of G with $G_1 = G/H$ of finite type. If B is a simple G-algebra then B^H is a simple G_1-algebra, because $I = (BI)^H$ for a G_1-ideal I of B^H. Thus $\mathrm{MOD}(G_1, C^H)$ is semisimple by Lemma 4.2, so every (G_1, C^H)-module is a direct summand of a free (G_1, C^H)-module $C^H \otimes_k M$, with $M \in \mathrm{REP}_k(G_1)$. By the isomorphisms $\mathrm{Hom}_{G_1}(M, C^H \otimes_k N) \simeq \mathrm{Hom}_G(M, C \otimes_k N)$ for $M, N \in \mathrm{REP}_k(G_1)$, the k-tensor functor $F = C \otimes_{C^H} - : \mathrm{MOD}(G_1, C^H) \to \mathrm{MOD}(G, C)$ is thus fully faithful. Further by semisimplicity of $\mathrm{MOD}(G, C)$, every (G, C)-submodule of one in the essential image of F is in the essential image of F. Thus FC_1 is a simple G-algebra if C_1 is a simple G_1-algebra over C^H, and $FC_1 \neq C$ if $C_1 \neq C^H$. Since an embedding $C_1 \to A^H \subset A$ over C^H defines an embedding $FC_1 \to A$ over C, it follows that C^H is a maximal simple G_1-subalgebra of A^H. Applying (2) with G_1, A^H, $\bar{A}^H$ and C^H for G, A, $\bar{A}$ and D then shows that $C^H \to \bar{A}^H$ is an isomorphism. □

6. Proof of Theorem 1.1

In the category of k-tensor categories and tensor isomorphism classes of k-tensor functors, the coproduct of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the pseudo-Abelian hull $\mathcal{C}_1 \otimes_k \mathcal{C}_2$ of the category with objects $\mathrm{Ob}\,\mathcal{C}_1 \times \mathrm{Ob}\,\mathcal{C}_1$, hom k-spaces tensor products of those of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a suitable tensor structure. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are semisimple positive with $\mathrm{End}(\mathbf{1}) = k$, so is $\mathcal{C}_1 \otimes_k \mathcal{C}_2$: its endomorphism algebras are semisimple by Lemma 4.1, and it is generated by objects in the image of the $\mathcal{C}_i \to \mathcal{C}_1 \otimes_k \mathcal{C}_2$, and so is positive by Lemmas 3.1 and 3.2.

Lemma 6.1. *Let $\mathcal{D}$ and A be as in Lemma* 4.2. *Then A has a unique simple quotient $\bar{A}$, and the projection $A \to \bar{A}$ has a right inverse in the category of algebras in* $\mathrm{Ind}(\mathcal{D})$.

Proof. By Lemma 3.4 we may assume $\mathcal{D} = \mathrm{Mod}(G, D)$ with G proreductive and $D^G = k$. Then D is G-simple and $\mathrm{Ind}(\mathcal{D}) = \mathrm{MOD}(G, D)$, by Lemma 4.2. The result thus follows from Lemma 5.3. □

Lemma 6.2. *If $\mathcal{A}$ and $\mathcal{D}$ are as in Theorem* 1.1, *then every k-tensor functor $\mathcal{D} \to \mathcal{A}$ factors, up to tensor isomorphism, through a right quasi-inverse to the projection $\mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$.*

Proof. By Lemmas 3.1 and 3.2 there is an essentially surjective k-tensor functor $\mathrm{Rep}_k(G) \to \mathcal{A}$, with G a product of general linear groups. Since $\mathcal{D} \to \mathcal{A}$ factors up to tensor isomorphism through $\mathcal{D} \otimes_k \mathrm{Rep}_k(G) \to \mathcal{A}$, we may by replacing $\mathcal{D}$ with $\mathcal{D} \otimes_k \mathrm{Rep}_k(G)$ assume that $\mathcal{D} \to \mathcal{A}$ is essentially surjective. Applying Lemma 3.3, we may then suppose that $\mathcal{A} = \mathrm{Mod}(A)$ for an algebra A in $\mathrm{Ind}(\mathcal{D})$, and that $\mathcal{D} \to \mathcal{A}$ is $A \otimes -$.

By Lemma 6.1 A has a simple quotient $\bar{A}$, and by Lemma 4.2 $\mathrm{Mod}(\bar{A})$ is semisimple. Since $\bar{A} \otimes_A - : \mathrm{Mod}(A) \to \mathrm{Mod}(\bar{A})$ is full, it factors through a k-tensor equivalence $\mathrm{Mod}(A)/\mathcal{R}(\mathrm{Mod}(A)) \to \mathrm{Mod}(\bar{A})$, by Lemma 4.1. It now suffices to note that if $\bar{A} \to A$ is right inverse to $A \to \bar{A}$ as in Lemma 6.1, then $A \otimes_{\bar{A}} - : \mathrm{Mod}(\bar{A}) \to \mathrm{Mod}(A)$ is right quasi-inverse to $\bar{A} \otimes_A -$, and $A \otimes - : \mathcal{D} \to \mathrm{Mod}(A)$ factors up to tensor isomorphism as $(A \otimes_{\bar{A}} -) \circ (\bar{A} \otimes -)$. □

To prove Theorem 1.1, it remains after Lemmas 4.1 and 6.2 only to show that any two right quasi-inverses T_1 and T_2 to the projection $P : \mathcal{A} \to \mathcal{A}/\mathcal{R}(\mathcal{A})$ are tensor isomorphic. If $\simeq$ denotes tensor isomorphism, then $T_i \simeq T'T_i'$, with $T' : \mathcal{A}/\mathcal{R}(\mathcal{A}) \otimes_k \mathcal{A}/\mathcal{R}(\mathcal{A}) \to \mathcal{A}$. By Lemma 6.2, $T' \simeq TS$ for some T with $PT \simeq \mathrm{Id}$. Then $S \simeq PT'$, so $T_i \simeq TST_i' \simeq TPT_i \simeq T$ and $T_1 \simeq T_2$.

7. Structure theorems

If G is a proreductive k-group and A is a G-algebra, then $\mathrm{Mod}(G, A)$ is k-tensor equivalent to the category $\mathrm{Vec}(G, X)$ of G-equivariant vector bundles over $X = \mathrm{Spec}(A)$, since each such bundle is a direct summand of the pullback along $X \to \mathrm{Spec}(k)$ of a representation of G. Thus by Lemma 3.4, any positive k-tensor category is k-tensor equivalent to $\mathrm{Vec}(G, X)$ for some proreductive G and affine G-scheme X.

Let $\mathcal{A}$ be a positive k-tensor category with $\mathrm{End}_{\mathcal{A}}(\mathbf{1}) = k$, and suppose that k is algebraically closed. It can be shown by applying Lemma 3.3 to the right quasi-inverse of Theorem 1.1 that, among pairs (G, X) with G proreductive, X affine and such that $\mathrm{Vec}(G, X)$ is k-tensor equivalent to a $\mathcal{A}$, there is one (G_0, X_0) for which X_0 has a k-point fixed by G_0. Further for any other such (G, X) there is an embedding $G_0 \to G$ of k-groups such that X is G-isomorphic to the homogeneous fibre space $G *_{G_0} X_0$.

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