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Dynamical Systems/Ordinary Differential Equations

On the Hopf bifurcation for flows

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Abstract

Under fairly general hypotheses, we prove the existence of the families of periodic orbits obtained by Hopf bifurcation, with emphasis on their smoothness. A Banach version of a theorem of Lyapounov is obtained as a corollary. The proofs are complete, simple and original. ***To cite this article: M. Chaperon, S. López de Medrano, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

La bifurcation de Hopf pour les flots. Sous des hypothèses très générales, nous prouvons l'existence des familles d'orbites périodiques obtenues par bifurcation de Hopf, en insistant sur leur régularité. Nous en déduisons une version banachique d'un théorème de Lyapounov. Les démonstrations sont complètes, simples et originales. ***Pour citer cet article : M. Chaperon, S. López de Medrano, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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Hypothèses et préliminaires

On pose $k=\infty$ ou $k=\omega$, et C^ω signifie « analytique réel ». On se donne deux variétés banachiques U, V et une famille locale C^k de champs de vecteurs sur V dépendant du paramètre $u\in U$, c'est-à-dire une application $X:(U\times V,(u_0,x_0))\to TV$ de classe C^k telle que chaque $X_u:x\mapsto X(u,x)$ soit un champ de vecteurs sur V. On note $\widetilde{X}(u,x):=(0,X_u(x))$ le *déploiement associé à* X. On fait les hypothèses suivantes :

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H1 On a $X_{u_0}(x_0) = 0$; autrement dit, (u_0, x_0) est un zéro de $\widetilde{X}$. On pose $E := T_{x_0}V$ et $L_0 := \mathrm{d}_{x_0}X_{u_0} : E \to E$.
H2 L'endomorphisme L_0 admet une valeur propre simple imaginaire pure $2\pi\mathrm{i}\nu_0$ avec $\nu_0 > 0$: le sous-espace propre associé du complexifié $L_{0,\mathbf{C}}$ est une droite complexe D, et $2\pi\mathrm{i}\nu_0$ n'est pas dans le spectre de l'endomorphisme de $E_{\mathbf{C}}/D$ induit par $L_{0,\mathbf{C}}$.
H3 Par conséquent, si $P \subset E$ désigne le 2-plan L_0-invariant engendré par les parties réelles et imaginaires des vecteurs propres de $L_{0,\mathbf{C}}$ associés à $2\pi\mathrm{i}\nu_0$, le spectre $\operatorname{Spec}\dot{L}_0$ du complexifié de l'endomorphisme $\dot{L}_0$ de E/P induit by L_0 ne contient ni $2\pi\mathrm{i}\nu_0$, ni $-2\pi\mathrm{i}\nu_0$. *On suppose en outre* que $\operatorname{Spec}\dot{L}_0$ ne contient aucun multiple entier de $2\pi\mathrm{i}\nu_0$. On a donc $0 \notin \operatorname{Spec} L_0$, d'où, par le théorème des fonctions implicites, la

Proposition. *Au voisinage de (u_0, x_0), les zéros de $\widetilde{X}$ forment le graphe $x = \varphi(u)$ d'une application φ de classe C^k. Il existe un unique germe d'application continue $\alpha : (U, u_0) \to \mathbf{C}$ tel que $\alpha(u)$ soit valeur propre de $\mathrm{d}_{\varphi(u)}X_u$ et que $\alpha(u_0) = 2\pi\mathrm{i}\nu_0$; de plus, α est de classe C^k.*

Nous pouvons énoncer notre dernière hypothèse :

H4 La différentielle $\mathrm{d}_{u_0}\Re\alpha : T_{u_0}U \to \mathbf{R}$ n'est pas nulle.

En désignant par h_u^t le flot de X_u, notre résultat porte sur les *points périodiques* des X_u, c'est-à-dire sur les $x \in V$ tels qu'il existe $T > 0$ vérifiant $h_u^T(x) = x$; il est commode d'exprimer cela par le fait que (u, x) est un point périodique de période T de $\widetilde{X}$. On note $T_0 := 1/\nu_0$ la période primitive de la restriction de e^{tL_0} au plan P.

Théorème. *Sous ces hypothèses, les propriétés suivantes sont vérifiées* :

(i) *Les points périodiques de $\widetilde{X}$ de période voisine de T_0 forment, près de (u_0, x_0), la réunion de deux sous-variétés C^k invariantes par le flot de $\widetilde{X}$: d'une part le graphe de φ, ensemble des zéros de $\widetilde{X}$ et, d'autre part, une sous-variété W_1 d'espace tangent $T_{(u_0,x_0)}W_1 = \{(\delta u, \delta x + \mathrm{d}_{u_0}\varphi(\delta u)) : (\delta u, \delta x) \in \operatorname{Ker}(\mathrm{d}_{u_0}\Re\alpha) \times P\}$.*

(ii) *L'intersection $W_0 \cap W_1$ est la sous-variété C^k formée des $(u, \varphi(u))$ avec $\Re\alpha(u) = 0$. Elle est de codimension* 1 *dans W_0 et de codimension* 2 *dans W_1, de sorte que l'ouvert $W_1 \setminus W_0$ de W_1 formé de points « vraiment périodiques » de $\widetilde{X}$ rencontre tout voisinage de (u_0, x_0).*

(iii) *Pour les points $(u, x) \in W_1 \setminus W_0$, la période $T(u, x)$ voisine de T_0 considérée dans* (i) *est en fait la période primitive. La fonction T ainsi définie se prolonge par continuité en une fonction C^k sur W_1 vérifiant $T(u, \varphi(u)) = 2\pi\mathrm{i}/\alpha(u)$.*

Apparaît donc en (u_0, x_0) une famille à $\dim(U)$ paramètres d'orbites périodiques de $\widetilde{X}$ de période primitive voisine de T_0, et la réunion (*disjointe*) *de ces orbites périodiques est $W_1 \setminus W_0$.*

Suivant une idée d'Alexander et Yorke, on en déduit une version banachique d'un théorème de Lyapounov, dont voici les hypothèses. On suppose V munie d'une *forme symplectique C^k*, c'est-à-dire d'une 2-forme fermée ω telle que, pour tout $x \in V$, l'application $T_xV \ni \mathbf{v} \mapsto \omega_x\mathbf{v}$ (produit intérieur) soit un isomorphisme de T_xV sur son dual. On désigne par X_0 un champ de vecteurs C^k sur V tel que le produit intérieur ωX_0 soit une 1-forme fermée (« champ localement hamiltonien ») et l'on considère un zéro x_0 de X_0 où les hypothèses suivantes sont satisfaites :

H2′ L'endomorphisme $L_0 := \mathrm{d}_{x_0}X_0$ de $E := T_{x_0}V$ admet une valeur propre simple (au sens de H2) imaginaire pure $2\pi\mathrm{i}\nu_0$ avec $\nu_0 > 0$.
H3′ Si $P \subset E$ désigne le 2-plan L_0-invariant engendré par les parties réelles et imaginaires des vecteurs propres du complexifié de L_0 associés à $2\pi\mathrm{i}\nu_0$, le spectre du complexifié de l'endomorphisme de E/P induit by L_0 ne contient aucun multiple entier de $2\pi\mathrm{i}\nu_0$. En particulier, L_0 est inversible, donc le zéro x_0 de X_0 est isolé.

Corollaire (Lyapounov). *Sous ces hypothèses, les propriétés suivantes sont vérifiées* :

(i) *Les points périodiques de X_0 de période voisine de $T_0 := 1/\nu_0$ forment, au voisinage de x_0, une surface W de classe C^k passant par x_0, invariante par le flot de X_0 et telle que $T_{x_0}W = P$. L'ouvert $W \setminus \{x_0\}$ de W formé de points « vraiment périodiques » de X_0 rencontre donc tout voisinage de x_0.*

(ii) *Pour les points $x \in W \setminus \{x_0\}$, la période $T(x)$ voisine de T_0 considérée dans* (i) *est en fait la période primitive. La fonction T ainsi définie se prolonge en une fonction C^k sur W telle que $T(x_0) = T_0$.*

Apparaît donc en x_0 une famille à un paramètre d'orbites périodiques de X_0 de période primitive voisine de T_0, et leur réunion (*disjointe*) *est $W \setminus \{x_0\}$. Chacune d'entre elles entoure x_0 dans W.*

1. Hypotheses and preliminaries

We let $k = \infty$ or $k = \omega$, and C^ω means 'real analytic'. Consider two Banach manifolds U, V and a local C^k family of vector fields on V with parameter $u \in U$, i.e. a C^k map $X : (U \times V, (u_0, x_0)) \to TV$ such that each $X_u : x \mapsto X(u, x)$ is a vector field. Let $\widetilde{X}(u, x) := (0, h_u(x))$ denote the *unfolding associated to X*. Assume the following:

H1 We have $X_{u_0}(x_0) = x_0$; in other words, (u_0, x_0) is a zero of $\widetilde{X}$. Let $E := T_{x_0}V$ and $L_0 := \mathrm{d}_{x_0}X_{u_0} : E \to E$.

H2 The endomorphism L_0 has a simple purely imaginary eigenvalue $2\pi \mathrm{i}\nu_0$ with $\nu_0 > 0$: the associated eigenspace of the complexified endomorphism $L_{0,\mathbf{C}}$ is a complex line D of the complexified space $E_\mathbf{C}$, and $2\pi \mathrm{i}\nu_0$ does not lie in the spectrum of the endomorphism of $E_\mathbf{C}/D$ induced by $L_{0,\mathbf{C}}$.

H3 Therefore, if $P \subset E$ denotes the L_0-invariant 2-plane generated by the real and imaginary parts of the eigenvectors of $L_{0,\mathbf{C}}$ associated to $2\pi \mathrm{i}\nu_0$, the spectrum $\mathrm{Spec}\, \dot{L}_0$ of the complexified of the endomorphism $\dot{L}_0$ of E/P induced by L_0 contains neither $2\pi \mathrm{i}\nu_0$, nor $-2\pi \mathrm{i}\nu_0$. *Assume that, moreover,* $\mathrm{Spec}\, \dot{L}_0$ *does not contain any integer multiple of* $2\pi \mathrm{i}\nu_0$. In particular, we have $0 \notin \mathrm{Spec}\, L_0$, hence, by the implicit function theorem:

Proposition. *Near (u_0, x_0), the zeros of $\widetilde{X}$ form the graph $x = \varphi(u)$ of a C^k map φ. There exists a unique continuous map germ $\alpha : (U, u_0) \to \mathbf{C}$ such that $\alpha(u)$ is an eigenvalue of $\mathrm{d}_{\varphi(u)}X_u$ and that $\alpha(u_0) = 2\pi \mathrm{i}\nu_0$; moreover, α is C^k.*

We can now state our last hypothesis:

H4 The differential $\mathrm{d}_{u_0}\Re\alpha : T_{u_0}U \to \mathbf{R}$ is nonzero.

Denoting by h_u^t the flow of X_u, the *periodic points* of X_u are those $x \in V$ such that $h_u^T(x) = x$ for some positive T. This amounts to saying that (u, x) is a T-periodic point of $\widetilde{X}$. Let $T_0 = 1/\nu_0$ be the primitive period of e^{tL_0} restricted to P. Now comes our main result:

Theorem. *Under the above hypotheses, we have the following*:

(i) *Near (u_0, x_0), the periodic points of $\widetilde{X}$ whose period is close to T_0 form the union of two C^k submanifolds invariant by the flow of $\widetilde{X}$: the set of zeros of $\widetilde{X}$, i.e. the graph W_0 of φ, and a submanifold W_1 with $T_{(u_0,x_0)}W_1 = \{(\delta u, \delta x + \mathrm{d}_{u_0}\varphi(\delta u))\colon (\delta u, \delta x) \in \mathrm{Ker}(\mathrm{d}_{u_0}\Re\alpha) \times P\}$.*

(ii) *The intersection $W_0 \cap W_1$ is the C^k submanifold consisting of those $(u, \varphi(u))$ with $\Re\alpha(u) = 0$. It has codimension* 1 *in W_0 and* 2 *in W_1, so that the open subset $W_1 \setminus W_0$ of W_1 consisting of 'truly periodic' points of $\widetilde{X}$ intersects every neighbourhood of (u_0, x_0).*

(iii) *For $(u, x) \in W_1 \setminus W_0$, the period $T(u, x)$ close to T_0 considered in* (i) *is in fact the primitive period. The function T so defined extends by continuity to a C^k function on W_1 which satisfies $T(u, \varphi(u)) = 2\pi \mathrm{i}/\alpha(u)$.*

Thus, there appears at (u_0, x_0) a $\dim(U)$-parameter family of periodic orbits of $\widetilde{X}$ whose primitive period is close to T_0, and their (*disjoint*) *union is $W_1 \setminus W_0$.*

Following an idea of Alexander and Yorke, we shall deduce from this a Banach version of a theorem by Lyapounov, whose hypotheses we now state. We assume V endowed with a C^k *symplectic form*, i.e. a closed differential 2-form ω such that, for every $x \in V$, the map $T_xV \ni \mathbf{v} \mapsto \omega_x\mathbf{v}$ (interior product) is an isomorphism of T_xV onto its dual space. We denote by X_0 a C^k vector field on V such that the interior product ωX_0 is a closed 1-form ('locally Hamiltonian vector field') and we consider a zero x_0 of X_0 satisfying the following hypotheses:

H2′ The endomorphism $L_0 := \mathrm{d}_{x_0}X_0$ of $E := T_{x_0}V$ admit a simple (in the sense of H2) purely imaginary eigenvalue $2\pi \mathrm{i}\nu_0$ with $\nu_0 > 0$.

H3′ Denoting by $P \subset E$ the L_0-invariant 2-plane generated by the real and imaginary parts of the eigenvectors of $L_{0,\mathbf{C}}$ associated to $2\pi \mathrm{i}\nu_0$, the spectrum of the complexified map of the endomorphism of E/P induced by L_0 does not contain any integer multiple of $2\pi \mathrm{i}\nu_0$. Thus, L_0 is invertible and therefore x_0 is an isolated zero of X_0.

Corollary (Lyapounov). *Under these hypotheses, we have the following*:

(i) *Near x_0, the periodic points of X_0 whose period is close to $T_0 := 1/\nu_0$ form a C^k surface W passing through x_0, invariant by the flow of X_0 and such that $T_{x_0}W = P$. The open subset $W \setminus \{x_0\}$ of W consisting of 'truly periodic' points of X_0 therefore intersects every neighbourhood of x_0.*

(ii) *For $x \in W \setminus \{x_0\}$, the period $T(x)$ close to T_0 considered in (i) is in fact the primitive period. The function T so defined extends to a C^k function on W such that $T(x_0) = T_0$.*

Thus, there appears at x_0 a one parameter family of periodic orbits of X_0 whose primitive period is close to T_0, and their (*disjoint*) *union is $W \setminus \{x_0\}$. Each bounds in W a disk containing x_0.*

2. Proof of the theorem

Taking charts, we may assume $V = E$, $U = T_{u_0}U$ and $(u_0, v_0) = (0, 0)$. The first part of the proposition follows from the implicit function theorem applied to the equation $X_u(x) = 0$. The change of variables $(u, x) \mapsto (u, x - \varphi(u))$ enables us to assume that $\varphi = 0$, i.e. $X_u(0) = 0$ for every u.

Proof of the rest of the proposition. With the notation of H1, choose any nonzero $\mathbf{v}_0 \in D$, any closed complementary complex subspace K of D in $E_\mathbf{C}$, and identify $E_\mathbf{C}$ to $\mathbf{C} \times K$ by the isomorphism $(z, \mathbf{w}) \mapsto z\mathbf{v}_0 + \mathbf{w}$. Then, $L_{0,\mathbf{C}}$ takes the form $\left(\begin{smallmatrix} 2\pi \mathrm{i}\nu_0 & b_0 \\ 0 & \mathrm{d}_0 \end{smallmatrix}\right)$, where $\mathrm{d}_0 : K \to K$ is a realisation of the endomorphism of $E_\mathbf{C}/D$ induced by $L_{0,\mathbf{C}}$ and therefore does not have $2\pi \mathrm{i}\nu_0$ in its spectrum. We can apply the implicit function theorem to the (complex) polynomial equation $F(L, \lambda, \mathbf{w}) := L(1, \mathbf{w}) - \lambda(1, \mathbf{w}) = 0$ at $(L_0, 2\pi \mathrm{i}\nu_0, 0) \in L(E_\mathbf{C}) \times \mathbf{C} \times K$; indeed, the partial $(\delta\lambda, \delta\mathbf{w}) \mapsto (b_0\delta\mathbf{w} - \delta\lambda, (\mathrm{d}_0 - 2\pi \mathrm{i}\nu_0 \,\mathrm{Id}_K)\delta\mathbf{w})$ of F with respect to $(\lambda, \mathbf{w})$ at $(L_0, 2\pi \mathrm{i}\nu_0, 0)$ is an automorphism because so is $\mathrm{d}_0 - 2\pi \mathrm{i}\nu_0 \,\mathrm{Id}_K$. Denoting the (holomorphic) implicit function by $(\lambda(L), \mathbf{w}(L))$, we get $\alpha(u) = \lambda((\mathrm{d}_{\varphi(u)}X_u)_\mathbf{C})$ and also the associated eigenvector $\mathbf{v}(u) := \mathbf{v}_0 + \mathbf{w}((\mathrm{d}_{\varphi(u)}X_u)_\mathbf{C})$ of $(\mathrm{d}_{\varphi(u)}X_u)_\mathbf{C} = DX_u(0)_\mathbf{C}$, a C^k function of u too. □

Proof of the theorem itself. We first establish a diagonalization result:

Lemma 2.1. *One can assume that the family X_u satisfies, in a decomposition $E = \mathbf{C} \times H$, the condition $DX_u(0) = \left(\begin{smallmatrix} \alpha(u) & 0 \\ 0 & d(u) \end{smallmatrix}\right)$, where $d : (U, u_0) \to L(H, H)$ is C^k.*

Proof. With the notation of the proof of the proposition, let H be any closed complementary subspace of the 2-plane P generated by $\Re\mathbf{v}(0)$ and $\Im\mathbf{v}(0)$. For u close to $0 = u_0$, the mapping $\mathbf{C} \times H \ni (x + \mathrm{i}y, \mathbf{w}) \mapsto x\Re\mathbf{v}(u) - y\Im\mathbf{v}(u) + \mathbf{w} \in E$ is an isomorphism depending C^k on u and conjugating $DX_u(0)$ to an endomorphism of $\mathbf{C} \times H$ of the form $\left(\begin{smallmatrix} \alpha(u) & b(u) \\ 0 & d(u) \end{smallmatrix}\right)$. Now, we can kill $b(u)$ by the variable change $(z, \mathbf{w}) \mapsto (z + c(u)\mathbf{w}, \mathbf{w})$, $c(u) \in L(H, \mathbf{C})$, provided $\alpha(u)c(u) - c(u) \circ d(u) = b(u)$; as the hypothesis $\operatorname{Spec} d(0) \not\ni 2\pi \mathrm{i}\nu_0$ implies that the spectrum $\operatorname{Spec} d(u)$ of the complex endomorphism ${}^t d(u) : c \mapsto c \circ d(u)$ of $L(H, \mathbf{C})$ does not contain $\alpha(u)$ for small u, this equation has the unique solution $c(u) = (\alpha(u)\,\mathrm{Id}_{L(H,\mathbf{C})} - {}^t d(u))^{-1}b(u)$, a C^k function of u. Composing our two variable changes, we do get a C^k family of isomorphisms $Q(u):\mathbf{C}\times H\to E$ such that $Q(u)^{-1}DX_u(0)Q(u)=\begin{pmatrix}\alpha(u)&0\\0&d(u)\end{pmatrix}$. If we replace X_u by $Q(u)^{-1}\circ X_u\circ Q(u)$, we get Lemma 2.1. □

Denoting by f_u^t, g_u^t the components of h_u^t in this decomposition $E=\mathbf{C}\times H$, we have to solve the C^k system

$$f_u^t(z,w)=z \quad\text{and}\quad g_u^t(z,w)=w$$

near $(t,u,z,w)=(T_0,0,0,0)$. As H2 implies that $\partial_w g_0^{T_0}(0,0)=\mathrm{e}^{T_0 d(0)}$ does not have 1 in its spectrum, the second equation defines a C^k implicit function $w=W(t,u,z)$ satisfying $W(t,u,0)=0$ and (since $\partial_z g_u^t(0,0)=0$) $\partial_z W(t,u,0)=0$. Thus, we should solve near $(t,u,z)=(T_0,0,0)$

$$f_u^t\big(z,W(t,u,z)\big)=z,\quad (t,u,z)\in\mathbf{R}\times U\times\mathbf{C},\tag{1}$$

whose spurious solution $z=0$ gives the fixed points.

Lemma 2.2. *Fix a decomposition $U=\mathbf{R}\times U_0$, $u=(\mu,\nu)$, such that $\partial_\nu\Re\alpha(0)=0$ and therefore $\partial_\mu\Re\alpha(0)\neq 0$. Then, we have the following*:

(i) *The solutions of* (1) *near* $(T_0,0,0)$ *form the union of* $\{z=0\}$ *and the graph* $(t,\mu)=(\tau(\nu,z),M(\nu,z))$ *of a continuous function, C^k in* $\{z\neq 0\}$.

(ii) *Therefore, near $(u_0,x_0)=(0,0)$, the manifold W_1 is given by $(\mu,w)=(M(\nu,z),\underline{W}(\nu,z))$, where $\underline{W}(\nu,z):=W(\tau(\nu,z),M(\nu,z),\nu,z)$ is continuous, C^k in $\{z\neq 0\}$, and the function $T:(M(\nu,z),\nu,z,\underline{W}(\nu,z))\mapsto\tau(\nu,z)$ is continuous, C^k in $W_1\setminus W_0$.*

(iii) *All the statements of the theorem are true, except perhaps smoothness.*

Proof. In polar coordinates $z=r\,\mathrm{e}^{\mathrm{i}\theta}$, we have $f_u^t(r\,\mathrm{e}^{\mathrm{i}\theta},W(t,u,r\,\mathrm{e}^{\mathrm{i}\theta}))=r\,\mathrm{e}^{\mathrm{i}\theta}F(t,u,r,\theta)$, where F is a C^k complex function. By Lemma 2.1, it satisfies $F(t,u,0,\theta)=\mathrm{e}^{-\mathrm{i}\theta}\partial_z f_u^t(0,0)\mathrm{e}^{\mathrm{i}\theta}=\mathrm{e}^{\alpha(u)t}$ since $\partial_z W(t,u,0)=0$. After factoring out the fixed points $r\,\mathrm{e}^{\mathrm{i}\theta}=0$, Eq. (1) becomes

$$F(t,u,r,\theta)=1.\tag{2}$$

Now, as $F(t,u,0,\theta)=\mathrm{e}^{\alpha(u)t}$, we have that $\partial_{(t,\mu)}F(T_0,0,0,\theta)=2\pi\mathrm{i}\nu_0\,\mathrm{d}t+T_0\partial_\mu\alpha(0)\,\mathrm{d}\mu$, which is an isomorphism since $\Re\partial_\mu\alpha(0)\neq 0$. Therefore, the solutions of (2) near $\{(t,u,r)=(T_0,0,0)\}$ are given by a C^k implicit function $(t,\mu)=(\tilde\tau(\nu,r,\theta),\widetilde{M}(\nu,r,\theta))$. As $F(t,u,0,\theta)=\mathrm{e}^{\alpha(u)t}$ is independent of θ, so are $\tilde\tau(\nu,0,\theta)$ and $\widetilde{M}(\nu,0,\theta)$, hence (i) with $\tau(\nu,r\,\mathrm{e}^{\mathrm{i}\theta})=\tilde\tau(\nu,r,\theta)$ and $M(\nu,r\,\mathrm{e}^{\mathrm{i}\theta})=\widetilde{M}(\nu,r,\theta)$. To prove (iii), first notice that W_0 is defined by $z=0$ and that $F(t,u,0,\theta)=1$ writes $\mathrm{e}^{\alpha(u)t}=1$, which is equivalent to $\Re\alpha(u)=0$ and $t=\tau(\nu,0)=2\pi\mathrm{i}m/\alpha(u)$ with $m\in\mathbf{Z}$. As τ is continuous and $\tau(0,0)=2\pi\mathrm{i}/\alpha(0)$, this forces $m=1$, hence part (ii) of the theorem and the second half of part (iii). Moreover, we do get the minimal period since T_0 is the smallest positive solution t of $\mathrm{e}^{\alpha(0)t}=1$. Let us prove that *if M is differentiable, then $DM(0,0)=0$*. Since $D\underline{W}(0,0)=0$, this will yield (after our coordinate changes) $T_{(0,0)}W_1=\{(\mu,w)=(0,0)\}$, hence the formula for $T_{(u_0,x_0)}W_1$ in the theorem. As $\mathrm{d}\widetilde{M}(0,0,\theta)=\partial_\nu M(0,0)\,\mathrm{d}\nu+\partial_z M(0,0)\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}r$, we should prove that $D\widetilde{M}(0,0,\theta)=0$ or, in other words, $\partial_{(\nu,r,\theta)}F(T_0,0,0,\theta)=0$. Since $\partial_{(\nu,\theta)}F(T_0,0,0,\theta)=0$, this reads $\partial_r F(T_0,0,0,\theta)=0$, i.e. $\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\partial_z^2 f_0^{T_0}(0)(\mathrm{e}^{\mathrm{i}\theta},\mathrm{e}^{\mathrm{i}\theta})=0$. Now, integrating the differential equation satisfied by $D^2h_0^t(0)$, we see that $\partial_z^2 f_0^t(0)(Z,Z)=\mathrm{e}^{t\alpha(0)}\int_0^t\mathrm{e}^{-\tau\alpha(0)}B(\mathrm{e}^{\tau\alpha(0)}Z,\mathrm{e}^{\tau\alpha(0)}Z)\,\mathrm{d}\tau$, where $B:\mathbf{C}^2\to\mathbf{C}$ is the first component of $\partial_z^2X_0(0)$. Writing $B(Z,Z)=aZ^2+bZ\overline{Z}+c\overline{Z}^2$, we do get $\partial_z^2 f_0^{T_0}(0)=0$. □

Lemma 2.3. *If (t,u,z) is a solution of* (1), *then z is a t-periodic point of the vector field $\zeta_{t,u}:z\mapsto Z_u(z,W(t,u,z))$, where $Z_u(z,w)\in\mathbf{C}$ denotes the first component of $X_u(z,w)\in\mathbf{C}\times H$.*

Proof. If $x=(z,w)$ is a t–periodic point of X_u, so is $h_u^s(x)$ for all s, hence $g_u^s(x)=W(t,u,f_u^s(x))$ and therefore $\frac{\mathrm{d}}{\mathrm{d}s}f_u^s(x)=Z_u(f_u^s(x),g_u^s(x))=Z_u(f_u^s(x),W(t,u,f_u^s(x)))$ for all s. This proves that $s\mapsto f_u^s(x)$ is the integral curve of $\zeta_{t,u}$ starting at z, which is t-periodic since $f_u^t(x)=z$. □

Lemma 2.4. *Let the decomposition* $U = \mathbf{R} \times U_0$, $u = (\mu, \nu)$, *be the same as in Lemma* 2.2. *Near* $(T_0, 0, 0) \in \mathbf{R} \times U \times \mathbf{C}$, *the set of those* (t, u, z) *such that* z *is a* t*-periodic point of* $\zeta_{t,u}$ *form the union of* $\{z = 0\}$ *and the graph* $(t, \mu) = (\tau_1(\nu, z), M_1(\nu, z))$ *of a* C^k *function. By Lemma* 2.2 *and Lemma* 2.3, *we must have* $\tau_1 = \tau$ *and* $M_1 = M$, *which shows that* W_1 *and* T *are* C^k *and completes the proof of the theorem.*

Proof if $k = \omega$**.** Using an analytic change of variables $(t, u, z) \mapsto (t, u, z - \varphi_{t,u}(\bar{z}))$ with $\varphi_{t,u}$ holomorphic and $\varphi_{t,u}(0) = \varphi'_{t,u}(0) = 0$, *we can assume that* $\zeta_{t,u}(z) = a_{t,u}(z)z$, *where* $a_{t,u}(z) \in \mathbf{C}$ *is an analytic function of* (t, u, z) *such that* $a_{t,u}(0) = \alpha(u)$. Indeed, each $\zeta_{t,u}$ can be complexified, yielding a holomorphic vector field $\zeta_{t,u,\mathbf{C}}$ near $0 \in \mathbf{C}^2$ whose linear part $D\zeta_{t,u,\mathbf{C}}(0)$ is $\alpha(u)v\partial_v + \overline{\alpha(u)}w\partial_w$ in the coordinates v, w extending $z, \bar{z}$. Therefore, the stable (resp. unstable) manifold of $i\zeta_{t,u,\mathbf{C}}$ at 0 is of the form $v = \varphi_{t,u}(w)$ (resp. $w = \overline{\varphi_{t,u}(\bar{v})}$). As these invariant manifolds are holomorphic, $\zeta_{t,u,\mathbf{C}}$ is tangent to them and therefore the change of variables $(v, w) \mapsto (v - \varphi_{t,u}(w), w - \overline{\varphi_{t,u}(\bar{v})})$ transforms $\zeta_{t,u,\mathbf{C}}$ into a vector field of the form $a_{t,u,\mathbf{C}}(v, w)v\partial_v + \overline{a_{t,u,\mathbf{C}}(\overline{w}, \bar{v})}w\partial_w$, hence what we claimed with $a_{t,u}(z) := a_{t,u,\mathbf{C}}(z, \bar{z})$.

In these new coordinates, the flow $\rho^s_{t,u}$ of $\zeta_{t,u}$ is of the form $\rho^s_{t,u}(z) = g(s, t, u, z)z$ with g complex-valued, analytic, and $g(s, t, u, 0) = \mathrm{e}^{\alpha(u)s}$. Therefore, after factoring z out, the equation $\rho^t_{t,u}(z) = z$ becomes $F_1(t, u, z) := g(t, t, u, z) = 1$, to which we can apply the implicit function theorem at $(t, u, z) = (T_0, 0, 0)$, since $\partial_{(t,\mu)} F_1(T_0, 0, 0) = \partial_{(t,\mu)}\, \mathrm{e}^{\alpha(u)t}|_{(t,u)=(T_0,0)}$ is an isomorphism (see the proof of Lemma 2.2). □

Proof if $k = \infty$**.** Using the same idea, let us prove that τ_1 and μ_1 are C^m for every positive integer m. By Taylor's formula, $\zeta_{u,t}(z) = \alpha(u)z + T_2 + \cdots + T_{m+1} + T_{m+2}$, where T_j is a homogeneous polynomial of degree j in $z, \bar{z}$ whose coefficients are C^∞ functions of t, u for $j \leqslant m + 1$ and C^∞ functions of t, u, z for $j = m + 2$. We can then make for $j = 2, \ldots, m + 1$ a C^∞ change of coordinates of the form $z \mapsto z + c_j(t, u)\bar{z}^j$ so that in the end the coefficient of $\bar{z}^j$ in T_j is 0 for $1 \leqslant j \leqslant m + 1$, hence $\zeta_{t,u}(z) = a_{t,u}(z)z$ with $a_{t,u}(z) := b(t, u, z) + c(t, u, z)\bar{z}^{m+2}/z$ and b, c of class C^∞. Now, $a_{t,u}(z)$ is C^m like $\bar{z}^{m+2}/z$. Therefore, the flow $\rho^s_{t,u}$ of the new $\zeta_{t,u}$ is of the form $\rho^s_{t,u}(z) = g(s, t, u, z)z$ with g of class C^m. Applying the implicit function theorem to the equation $F_1(t, u, z) := g(t, t, u, z) = 1$, we conclude (going back to the initial problem) that W_1 and T are C^m near (u_0, x_0). A priori, the domain around (u_0, x_0) where this is the case might shrink when $m \to \infty$. But we already know that $W_1 \setminus W_0$ and $T|_{W_1 \setminus W_0}$ are C^∞, and what we have just done applies at *any* point of $W_0 \cap W_1$ close to (u_0, x_0), not just (u_0, x_0), hence the theorem. □

Proof of the corollary. By Darboux's theorem, we may assume that $V = E$, that $x_0 = 0$ and that ω is a constant form. Therefore, the Lie derivative $\mathcal{L}_Y\omega = d(\omega Y)$ of ω with respect to $Y : x \mapsto x$ is $\mathcal{L}_Y\omega = 2\omega$; as $\mathcal{L}_{X_0}\omega = d(\omega X_0) = 0$, it follows that the *one*-parameter family $X_u := X_0 + uY$ satisfies $\mathcal{L}_{X_u}\omega = 2u\omega$. Since it also satisfies the hypotheses of our theorem, the t-periodic points of $\widetilde{X}$ with t close to T_0 form a C^k surface $W_1 \subset \mathbf{R} \times E$ through zero, with $T_0W_1 = \{0\} \times P$, so that we just have to show that W_1 is included in $\{u = 0\}$ near 0. Assume there are periodic orbits $\widetilde{C}_u = \{u\} \times C_u \subset W_1$ of $\widetilde{X}$ with $u \neq 0$ arbitrarily close to 0. Denote by T_u their period, by λ a primitive of ω and by h^t_u the flow of X_u. Since $\mathcal{L}_{X_u}\omega = 2u\omega$, we have $h^{t*}_u\omega = \mathrm{e}^{2ut}\omega$. For each disk $D \subset E$ bounded by C_u, it follows that $\int_D \omega = \int_{C_u} \lambda = \int_{C_u} h^{T_u*}_u\lambda = \int_D h^{T_u*}_u\omega = \mathrm{e}^{2uT_u}\int_D \omega$, hence $\int_{C_u} \lambda = 0$. Now, as $2\pi \mathrm{i}\nu_0$ is a simple eigenvalue of L_0, the plane P is symplectic and therefore, since $T_0W_1 = \{0\} \times P$, the pullback $\pi^*\omega$ of ω by the projection $\pi : \mathbf{R} \times E \to E$ induces a surface element σ on W_1 near 0. If $\widetilde{C}_u$ is close to 0, it bounds a disk $\widetilde{D}$ in W_1, hence the contradiction $0 = \int_{C_u} \lambda = \int_{\widetilde{C}_u} \pi^*\lambda = \int_{\widetilde{D}} \sigma \neq 0$. □

The method can also yield *finite* differentiability results [1].

References

[1] M. Chaperon, S. López de Medrano, J.L. Samaniego. On sub-harmonic bifurcations. C. R. Acad. Sci. Paris, Ser. I 304 (2005), in press.