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Corrigendum to the Note "Symplectic capacities of toric manifolds and combinatorial inequalities" [C. R. Acad. Sci. Paris, Ser. I 334 (10) (2002) 889–892]

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Abstract

In this Note we correct some results in Lu, Symplectic capacities of toric manifolds and combinatorial inequalities [C. R. Acad. Sci. Paris, Ser. I 334 (10) (2002) 889–892] on (pseudo) symplectic capacities for toric manifolds. ***To cite this article: G. Lu, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Capacités symplectiques de variétés toriques et des associés résultats. Daus cette Note, nous corrigeons des résultats associés dans Lu, Symplectic capacities of toric manifolds and combinatorial inequalities [C. R. Acad. Sci. Paris, Ser. I 334 (10) (2002) 889–892] sur les capacités (pseudo) symplectiques de variétés toriques. ***Pour citer cet article : G. Lu, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

The notion of the pseudo symplectic capacity was introduced by the author in [3,4]. ($\widehat{C}_{HZ}^{(2)}$ in [4] was written as $C_{HZ}^{(2\circ)}$ in the recent [3], v9 in view of some reader's suggestion.) In [4] three theorems were announced based on the author's work in [3] and Batyrev's computation for the quantum cohomology of the toric manifolds in [1]. However, Batyrev's results in [1] were true only for Fano toric manifolds. So our results in [4] can only hold for this class of manifolds. That is, Theorems 1, 2 in [4] should be written as:

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Theorem 1. *For a complete regular fan Σ in $\mathbb{R}^n$, let $G(\Sigma)=\{u_1,\dots,u_d\}$ be the set of all generators of 1-dimensional cones in Σ, and P_Σ be the compact toric manifold associated with Σ. Assume that φ is a strictly convex support function for Σ representing a Kähler form on P_Σ, and that $\Delta_\varphi=\{x\in(\mathbb{R}^n)^*\mid\langle x,m\rangle\geqslant-\varphi(m)\ \forall m\in\mathbb{R}^n\}$ is the corresponding Delzant polytope in $(\mathbb{R}^n)^*$. If P_Σ is also Fano, i.e., the anticanonical divisor $-K_{\mathrm{P}_\Sigma}$ is ample, then*

$$\Upsilon(\Sigma,\varphi):=\inf\left\{\sum_{k=1}^d\varphi(u_k)a_k>0\ \middle|\ \sum_{k=1}^d a_ku_k=0,\ a_k\in\mathbb{Z}_{\geqslant0},\ k=1,\dots,d\right\}>0, \tag{1}$$

and the Gromov width $\mathcal{W}_G$ and pseudo symplectic capacities $C=C^{(2)}_{HZ},C^{(2\circ)}_{HZ}$ satisfy:

$$\mathcal{W}_G(\mathrm{P}_\Sigma,\varphi)\leqslant C\big(\mathrm{P}_\Sigma,\varphi;pt,PD\big([\varphi]\big)\big)\leqslant\Upsilon(\Sigma,\varphi)\quad\forall n\geqslant2. \tag{2}$$

Moreover, whether P_Σ is Fano or not it always holds that

$$\mathcal{W}_G(\mathrm{P}_\Sigma,\varphi)\geqslant\frac{1}{2\pi}\mathcal{W}_G\big(\mathrm{Int}(\Delta_\varphi)\times\mathbb{T}^n,\omega_{\mathrm{can}}\big), \tag{3}$$

where $(\mathrm{Int}(\Delta_\varphi)\times\mathbb{T}^n,\omega_{\mathrm{can}})=(\{(x,\theta)\mid x\in\mathrm{Int}(\Delta_\varphi),\ \theta\in\mathbb{R}^n/2\pi\mathbb{Z}^n\},\ \sum_{k=1}^d\mathrm{d}x_k\wedge\mathrm{d}\theta_k)$.

If X_Δ is a Fano toric manifold associated with Delzant polytope in $(\mathbb{R}^n)^*$

$$\Delta=\bigcap_{k=1}^d\big\{x\in(\mathbb{R}^n)^*\ \big|\ l_k(x):=\langle x,u_k\rangle-\lambda_k\geqslant0\big\} \tag{4}$$

and ω_Δ is the canonical symplectic form on it then

$$\Upsilon(\Delta):=\inf\left\{-\sum_{k=1}^d\lambda_ka_k>0\ \middle|\ \sum_{k=1}^d a_ku_k=0,\ a_k\in\mathbb{Z}_{\geqslant0},\ k=1,\dots,d\right\}>0 \tag{5}$$

and it holds that for $C=C^{(2)}_{HZ},C^{(2\circ)}_{HZ}$ and any $n\geqslant2$,

$$\mathcal{W}_G(X_\Delta,\omega_\Delta)\leqslant C\big(X_\Delta,\omega_\Delta;pt,PD\big([\omega_\Delta]\big)\big)\leqslant2\pi\cdot\Upsilon(\Delta). \tag{6}$$

Furthermore, if $\mathrm{Vert}(\Delta)$ denotes the set of all vertices of Δ and $E_p(\Delta)$ is the shortest distance from the vertex p to the adjacent n vertexes, then for any capacity function c,

$$2\pi\cdot\max_{p\in\mathrm{Vert}(\Delta)}E_p(\Delta)\leqslant c(X_\Delta,\omega_\Delta) \tag{7}$$

whether X_Δ is Fano or not.

In general case we have:

Theorem 2. *Let P_Σ be the compact toric manifold associated with a complete regular fan Σ in $\mathbb{R}^n$ with $G(\Sigma)=\{u_1,\dots,u_d\}$. For a strictly convex support function φ (for Σ) representing a Kähler form on P_Σ let $\Lambda(\Sigma,\varphi)$ be the maximum of $\sum_{i=1}^d\varphi(u_i)a_i$ for which $(a_1,\dots,a_d)\in\mathbb{Z}^n_{\geqslant0}$ satisfies $\sum_{i=1}^d a_iu_i=0$ and $1\leqslant\sum_{i=1}^d a_i\leqslant n+1$. Then for $C=C^{(2)}_{HZ},C^{(2\circ)}_{HZ}$,*

$$0<\Lambda(\Sigma,\varphi)\leqslant(n+1)\max_i\varphi(u_i)\quad\text{and} \tag{8}$$

$$\mathcal{W}_G(\mathrm{P}_\Sigma,\varphi)\leqslant C\big(\mathrm{P}_\Sigma,\varphi;pt,PD\big([\varphi]\big)\big)\leqslant\Lambda(\Sigma,\varphi)\quad\forall n\geqslant2. \tag{9}$$

If (X_Δ,ω_Δ) is the toric manifold associated with the Delzant polytope $\Delta\subset(\mathbb{R}^n)^*$ in (4), but it might not be Fano, and $\Lambda(\Delta)$ $(=\Lambda(\Sigma_\Delta,\omega_\Delta))$ is the maximum of $-2\pi\sum_{i=1}^d\lambda_ia_i$ for all $(a_1,\dots,a_d)\in\mathbb{Z}^n_{\geqslant0}$ satisfying $\sum_{i=1}^d a_iu_i=0$ and $1\leqslant\sum_{i=1}^d a_i\leqslant n+1$, then

$$\Lambda(\Delta) \leqslant -2\pi(n+1)\min_i \lambda_i \quad \textit{and} \tag{10}$$

$$2\pi\mathcal{W}(\Delta) \leqslant \mathcal{W}_G(X_\Delta, \omega_\Delta) \leqslant C\bigl(X_\Delta, \omega_\Delta; pt, PD\bigl([\omega_\Delta]\bigr)\bigr) \leqslant \Lambda(\Delta) \tag{11}$$

for $C = C^{(2)}_{HZ}, C^{(2\circ)}_{HZ}$.

The projective toric manifolds are uniruled. The following proposition is a key to prove Theorem 2. Its proof may directly be obtained by combining Kollar's arguments in [2] and the proof of Proposition 7.3 in [3], v9, cf., [5].

Proposition 3. *For a uniruled manifold X of positive dimension n there exist homology classes $A \in H_2(X;\mathbb{Z})$ with $1 \leqslant c_1(A) \leqslant n+1$, $\alpha \in H_{2n-2}(X,\mathbb{Q})$ and $\beta \in H_*(X;\mathbb{Q})$ such that*

$$\Psi_{A,0,3}(pt; pt, \alpha, \beta) \neq 0. \tag{12}$$

In particular, this implies that there is a rational curve C with $0 < (-K_X \cdot C) \leqslant n+1$ through any general point of X.

The final claim in Proposition 3 was first proved by Mori for Fano manifolds. In general case Mori [6] told the author that it can be immediately obtained from Kollar's modification on a result in Proc. ICM90 by him. Our method as a consequence of (12) actually suggested possible further generalizations.

An outline of proof of Theorem 2. We only need to prove (8) and the second inequality in (9). (See [5] for the related details, notions and notations.) Since P_Σ is uniruled, Proposition 3 yields homology classes $A \in H_2(\mathrm{P}_\Sigma;\mathbb{Z})$ with $1 \leqslant c_1(A) \leqslant n+1$, $\alpha \in H_{2n-2}(\mathrm{P}_\Sigma,\mathbb{Q})$ and $\beta \in H_*(\mathrm{P}_\Sigma;\mathbb{Q})$ such that $\Psi_{A,0,3}(pt; pt, \alpha, \beta) \neq 0$. Since the Gromov–Witten invariants are deformation invariants it follows that $\langle[\varphi], A\rangle = \sum_{i=1}^d \varphi(u_i)\mu(A)_i > 0$ for any $\varphi \in K^\circ(\Sigma)$. Note that $K(\Sigma)$ is the closure of $K^\circ(\Sigma)$ in $H^2(\mathrm{P}_\Sigma,\mathbb{R})$. So $\langle[\psi], A\rangle = \sum_{i=1}^d \psi(u_i)\mu(A)_i \geqslant 0$ for any $\psi \in K(\Sigma)$. In particular we get that $\mu(A)_l = \sum_{i=1}^d \varphi_l(u_i)\mu(A)_i = \langle[\varphi_l], A\rangle \geqslant 0$, $l = 1, \ldots, d$. These show that A is very effective. By Theorem 2.1 in [5], $c_1(A) = \sum_{i=1}^d \mu(A)_i$ and thus $1 \leqslant \sum_{i=1}^d \mu(A)_i \leqslant n+1$. The definition of $\Lambda(\Sigma,\varphi)$ directly leads to

$$0 < \bigl\langle[\varphi], A\bigr\rangle = \sum_{i=1}^d \varphi(u_i)\mu(A)_i \leqslant \Lambda(\Sigma,\varphi).$$

By the definition of $\mathrm{GW}_0(M,\omega; pt, \alpha)$ in Definition 1.9 of [3], v9 we get that

$$\mathrm{GW}_0\bigl(\mathrm{P}_\Sigma, \varphi; pt, PD\bigl([\varphi]\bigr)\bigr) \leqslant \Lambda(\Sigma,\varphi).$$

Moreover it is clear that

$$\sum_{i=1}^d \varphi(u_i)\mu_i \leqslant \sum_{\varphi(u_i)>0} \varphi(u_i)\mu_i \leqslant (n+1)\max_i \varphi(u_i)$$

for each $\mu \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $\sum_{i=1}^d \mu_i u_i = 0$ and $1 \leqslant \sum_{i=1}^d \mu_i \leqslant n+1$. The desired results may be obtained from Theorem 1.13 in [3], v9. □

It is well-known that the blow-ups of a toric manifold at its toric fixed points are also toric manifolds. However, the blow up of a toric Fano manifold is not necessarily Fano again.

Theorem 4. *Let $\mathrm{P}_{\widetilde{\Sigma}}$ be a toric manifold obtained by a sequence of blowings up of a toric Fano manifold at toric fixed points. So $G(\Sigma) = \{u_1, \ldots, u_d\} \subset G(\widetilde{\Sigma})$. Then for any strictly convex support function φ for $\widetilde{\Sigma}$ (also strictly convex for Σ) it hold that*

$$\mathcal{W}_G(\mathrm{P}_{\widetilde{\Sigma}}, \varphi) \leqslant C\bigl(\mathrm{P}_{\widetilde{\Sigma}}, \varphi; pt, PD\bigl([\varphi]\bigr)\bigr) \leqslant 2\pi \cdot \Upsilon(\Sigma,\varphi)$$

for $C = C_{HZ}^{(2)}, C_{HZ}^{(2\circ)}$ *and any* $n \geqslant 2$. *Here* $\Upsilon(\Sigma, \varphi) > 0$ *is given by* (1) *though* $\Upsilon(\widetilde{\Sigma}, \varphi)$ *might equal to zero in the case* $\mathrm{P}_{\widetilde{\Sigma}}$ *is not Fano.*

A correction to Example (ii) in [4]. Let e_1^*, e_2^* and e_3^* be the dual basis of the standard basis e_1, e_2 and e_3 in $\mathbb{R}^3$, and Δ be a Delzant polytope with vertices $v_0 = 0$, $v_1 = e_1^*$, $v_2 = e_2^*$ and $v_3 = e_3^*$. It is well-known that the corresponding toric manifold $(X_\Delta, \omega_\Delta)$ is exactly $(\mathbb{CP}^3, 2\omega_{\mathrm{FS}})$, where the Fubini–Study ω_{FS} is assumed to satisfy $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} = \pi$. For $0 < a < 1$ consider a Delzant polytope $\Delta_a \subset (\mathbb{R}^3)^*$ with vertices $v_0 = 0$, $v_1 = e_1^*$, $v_2 = e_2^*$, $v_3 = ae_2^* + ae_3^*$, $v_4 = ae_3^*$, $v_5 = ae_1^* + ae_3^*$. Clearly, the normal vectors to the 2-dimensional faces of Δ_a are $u_1 = e_1^*$, $u_2 = e_2^*$, $u_3 = e_3^*$, $u_4 = -e_3^*$, $u_5 = -e_1^* - e_2^* - e_3^*$. So $\Delta_a = \bigcap_{k=1}^5 \{x \in (\mathbb{R}^3)^* \mid \langle x, u_k\rangle - \lambda_k \geqslant 0\}$, where $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = -a$, $\lambda_5 = -1$. The associated toric manifold $(X_{\Delta_a}, \omega_{\Delta_a})$ is exactly the blow-up of $(\mathbb{CP}^3, 2\omega_{\mathrm{FS}})$ of weight $2(1-a)$ at a fixed point of $\mathbb{T}^3$-action on $(\mathbb{CP}^3, 2\omega_{\mathrm{FS}})$. From Theorem 2 it follows that $\Lambda(\Delta_a) \leqslant -8\pi \min_i \lambda_i = 8\pi$ and

$$\mathcal{W}_G(X_{\Delta_a}, \omega_{\Delta_a}) \leqslant C\big(X_{\Delta_a}, \omega_{\Delta_a}; pt, PD\big([\omega_{\Delta_a}]\big)\big) \leqslant 8\pi \tag{13}$$

for $C = C_{HZ}^{(2)}, C_{HZ}^{(2\circ)}$. In particular, if $a = 1/2$ we can use Theorem 2.5 in [5] to check that $(X_{\Delta_{1/2}}, \omega_{\Delta_{1/2}})$ is Fano. By Theorem 1 we may get $\Upsilon(\Delta_{1/2}) = 1/2$ and (13) can be strengthen as:

$$\mathcal{W}_G(X_{\Delta_{1/2}}, \omega_{\Delta_{1/2}}) \leqslant C_{HZ}\big(X_{\Delta_{1/2}}, \omega_{\Delta_{1/2}}; pt, PD\big([\omega_{\Delta_{1/2}}]\big)\big) \leqslant \pi.$$

Theorem 4 in [4] should be *corrected* as:

Theorem 5. *Let* Σ *be a complete regular fan in* $\mathbb{R}^n$. *Then for any ample line bundle* $L \to \mathrm{P}_\Sigma$ *and any strictly convex support function* φ_L *representing the class* $c_1(L)$ *it holds that*

$$\varepsilon(L) \leqslant 2\pi \cdot \Lambda(\Sigma, \varphi_L)$$

and that $\varepsilon(L) \leqslant 2\pi \cdot \Upsilon(\Sigma, \varphi_L)$ *if* P_Σ *is also Fano. Here* $\varepsilon(L)$ *is the Seshadri constant of* L.

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