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Homological Algebra

Discrete Morse theory for free chain complexes

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Abstract

We extend the combinatorial Morse complex construction to arbitrary free chain complexes, and give a short, self-contained, and elementary proof of the quasi-isomorphism between the original chain complex and its Morse complex. Even stronger, the main result states that, if C_* is a free chain complex, and $\mathcal{M}$ an acyclic matching, then $C_* = C_*^{\mathcal{M}} \oplus T_*$, where $C_*^{\mathcal{M}}$ is the Morse complex generated by the critical elements, and T_* is an acyclic complex. ***To cite this article: D.N. Kozlov, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Résumé

Théorie de Morse pour des complexes de chaînes libres. On étend la construction du complexe de Morse combinatoire aux complexes de chaînes libres généraux, et on donne une démonstration brève et élémentaire du quasi-isomorphisme entre le complexe de chaînes original et son complexe de Morse. Plus précisément, le résultat principal dit que, si C_* est un complexe de chaînes libres et $\mathcal{M}$ est une correspondance acyclique, alors $C_* = C_*^{\mathcal{M}} \oplus T_*$, où $C_*^{\mathcal{M}}$ est le complexe de Morse engendré par les éléments critiques et T_* est un complexe acyclique. ***Pour citer cet article : D.N. Kozlov, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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La théorie de Morse discrète a été introduite par Forman [1], elle s'est révélée utile pour des calculs en combinatoire topologique. Il a été demontré dans [1, Theorem 8.2] que, étant donné une fonction de Morse discrète

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[1, Definition 2.1] sur un complexe CW fini K, le complexe de chaînes cellulaire $C_*(K;\mathbb{Z})$ est quasi-isomorphe au complexe de Morse combinatoire associé.

Dans cette Note, on étend cette construction au cas des complexes de chaînes libres généraux. On présente une démonstration indépendante et simple dans cette généralité, en particulier on arrive à une démonstration nouvelle et élémentaire des résultats de Forman. À un niveau plus élevé, on peut regarder notre démonstration comme un analogue algébrique des arguments présentés dans [2, Theorem 3.2].

Soit $\mathcal{R}$ un anneau commutatif général avec un élément neutre. On dit qu'un complexe de chaînes C_* de $\mathcal{R}$-modules $\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$ est *libre* si chaque C_n est un $\mathcal{R}$-module libre finiment engendré. Si les indices sont clairs, on écrit ∂ au lieux de ∂_n. On demande que C_* soit borné à droite.

Supposons qu'on ait choisit une base (un ensemble de générateurs libres) Ω_n pour chaque C_n. Dans ce cas, on dit qu'on a choisi une base $\Omega = \bigcup_n \Omega_n$ pour C_*, et on écrit (C_*, Ω) pour un complexe de chaînes avec une base. Un complexe de chaînes libre avec une base est l'objet principal de cette Note.

Définition 0.1. Soit (C_*, Ω) un complexe de chaînes libre avec une base.

(1) Une *correspondance partielle* $\mathcal{M} \subseteq \Omega \times \Omega$ sur (C_*, Ω) est une correspondance partielle sur le diagramme de Hasse de $P(C_*, \Omega)$, tel que si $b \succ a$ et b et a sont en correspondance, i.e., si $(a, b) \in M$, alors $w(b \succ a)$ est inversible.

(2) Une correspondance partielle sur (C_*, Ω) est *acyclique*, s'il n'y a pas de cycle

$$d(b_1) \prec b_1 \succ d(b_2) \prec b_2 \succ d(b_3) \prec \cdots \succ d(b_n) \prec b_n \succ d(b_1), \tag{1}$$

avec $n \geqslant 2$ et tous les $b_i \in \mathcal{U}(\Omega)$ différents.

Définition 0.2. Soit (C_*, Ω) un complexe de chaînes libre avec une base et soit $\mathcal{M}$ une correspondance acyclique. Le *complexe de Morse* $\cdots \xrightarrow{\partial^{\mathcal{M}}_{n+2}} C^{\mathcal{M}}_{n+1} \xrightarrow{\partial^{\mathcal{M}}_{n+1}} C^{\mathcal{M}}_n \xrightarrow{\partial^{\mathcal{M}}_n} C^{\mathcal{M}}_{n-1} \xrightarrow{\partial^{\mathcal{M}}_{n-1}} \cdots$ est défini comme suit. Le $\mathcal{R}$-module $C^{\mathcal{M}}_n$ est librement engendré par les elements de $\mathcal{C}_n(\Omega)$. L'opérateur de borne est défini par $\partial^{\mathcal{M}}_n(s) = \sum_p w(p) \cdot p_\bullet$ pour $s \in \mathcal{C}_n(\Omega)$, où la somme est prise sur tous les chemins alternés p qui satisfont $p^\bullet = s$.

Le complexe de chaînes $\cdots \longrightarrow 0 \longrightarrow \mathcal{R} \xrightarrow{\text{id}} \mathcal{R} \longrightarrow 0 \longrightarrow \cdots$ dans lequels les seuls modules non-triviaux se trouvent en dimension d et $d-1$, est appelé un *complexe de chaînes atomique*, qu'on écrit $\mathtt{Atom}(d)$.

Le résultat principal de cette Note est comme suit :

Théorème 0.3. *Supposons qu'on ait un complexe de chaînes libre avec une base* (C_*, Ω) *et une correspondance acyclique* $\mathcal{M}$*. Alors,* C_* *se décompose en une somme directe de complexes de chaînes* $C^{\mathcal{M}}_* \oplus T_*$ *et* $T_* \simeq \bigoplus_{(a,b)\in\mathcal{M}} \mathtt{Atom}(\dim b)$.

1. Acyclic matchings on chain complexes and the Morse complex

Discrete Morse theory was introduced by Forman, see [1], and it proved to be useful in various computations in topological combinatorics. It was shown, [1, Theorem 8.2], that, given a discrete Morse function, [1, Definition 2.1], on a finite CW complex K, the cellular chain complex $C_*(K;\mathbb{Z})$ is quasi-isomorphic to the associated combinatorial Morse complex.

In this Note, we extend this construction to the case of arbitrary free chain complexes. We give an independent, simple, and self-contained proof in this generality, in particular furnishing a new elementary and short derivation of Forman's result. On a higher level, our proof can be viewed as an algebraic analog of the argument given in [2, Theorem 3.2].

Let $\mathcal{R}$ be an arbitrary commutative ring with a unit. We say that a chain complex C_* consisting of $\mathcal{R}$-modules $\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$, is *free* if each C_n is a finitely generated free $\mathcal{R}$-module. When the indexing is clear, we simply write ∂ instead of ∂_n. We require C_* to be bounded on the right.

Assume that we have chosen a basis (i.e., a set of free generators) Ω_n for each C_n. In this case we say that we have chosen a basis $\Omega = \bigcup_n \Omega_n$ for C_*, and we write (C_*, Ω) to denote a chain complex with a basis. A free chain complex with a basis is the main character of this Note.

Given a free chain complex with a basis (C_*, Ω), and two elements $\alpha \in C_n$ and $b \in \Omega_n$, we denote the coefficient of b in the representation of α as a linear combination of the elements of Ω_n by $\mathfrak{K}_\Omega(\alpha, b)$, or, if the basis is clear, simply by $\mathfrak{K}(\alpha, b)$. For $x \in C_n$ we write $\dim x = n$. By convention, we set $\mathfrak{K}_\Omega(\alpha, b) = 0$ if the dimensions do not match, i.e., if $\dim \alpha \neq \dim b$.

Note that a free chain complex with a basis (C_*, Ω) can be represented as a ranked poset $P(C_*, \Omega)$, with $\mathcal{R}$-weights on the order relations. The elements of rank n correspond to the elements of Ω_n, and the weight of the covering relation $b \succ a$, for $b \in \Omega_n$, $a \in \Omega_{n-1}$, is simply defined by $w_\Omega(b \succ a) := \mathfrak{K}_\Omega(\partial b, a)$. In other words, $\partial b = \sum_{b \succ a} w_\Omega(b \succ a) a$, for each $b \in \Omega_n$. Again, if the basis is clear, we simply write $w(b \succ a)$.

Definition 1.1. Let (C_*, Ω) be a free chain complex with a basis. A *partial matching* $\mathcal{M} \subseteq \Omega \times \Omega$ on (C_*, Ω) is a partial matching on the covering graph of $P(C_*, \Omega)$, such that if $b \succ a$, and b and a are matched, i.e., if $(a, b) \in M$, then $w(b \succ a)$ is invertible.

Remark 1. Note that the Definition 1.1 is different from [1, Definition 2.1]. The latter is a topological definition, and has the condition that the matched cells form a regular pair (in the CW sense). In our algebraic setting it suffices to require the invertibility of the covering weight.

Given such a partial matching $\mathcal{M}$, we write $b = u(a)$, and $a = d(b)$, if $(a, b) \in \mathcal{M}$. We denote by $\mathcal{U}_n(\Omega)$ the set of all $b \in \Omega_n$, such that b is matched with some $a \in \Omega_{n-1}$, and, analogously, we denote by $\mathcal{D}_n(\Omega)$ the set of all $a \in \Omega_n$, which are matched with some $b \in \Omega_{n+1}$. We set $\mathcal{C}_n(\Omega) := \Omega_n \setminus \{\mathcal{U}_n(\Omega) \cup \mathcal{D}_n(\Omega)\}$ to be the set of all unmatched basis elements; these elements are called *critical*. Finally, we set $\mathcal{U}(\Omega) := \bigcup_n \mathcal{U}_n(\Omega)$, $\mathcal{D}(\Omega) := \bigcup_n \mathcal{D}_n(\Omega)$, and $\mathcal{C}(\Omega) := \bigcup_n \mathcal{C}_n(\Omega)$.

Given two basis elements $s \in \Omega_n$ and $t \in \Omega_{n-1}$, an *alternating path* is a sequence

$$p = \bigl(s \succ d(b_1) \prec b_1 \succ d(b_2) \prec b_2 \succ \cdots \succ d(b_n) \prec b_n \succ t\bigr), \tag{2}$$

where $n \geqslant 0$, and all $b_i \in \mathcal{U}(\Omega)$ are distinct. We use the notations $p^\bullet = s$ and $p_\bullet = t$. The *weight* of such an alternating path is defined to be the quotient

$$w(p) := (-1)^n \frac{w(s \succ d(b_1)) \cdot w(b_1 \succ d(b_2)) \cdot \cdots \cdot w(b_n \succ t)}{w(b_1 \succ d(b_1)) \cdot w(b_2 \succ d(b_2)) \cdot \cdots \cdot w(b_n \succ d(b_n))}.$$

Definition 1.2. A partial matching on (C_*, Ω) is called *acyclic*, if there does not exist a cycle

$$d(b_1) \prec b_1 \succ d(b_2) \prec b_2 \succ d(b_3) \prec \cdots \succ d(b_n) \prec b_n \succ d(b_1), \tag{3}$$

with $n \geqslant 2$, and all $b_i \in \mathcal{U}(\Omega)$ being distinct.

There is a nice alternative way to reformulate the notion of acyclic matching.

Proposition 1.3. *A partial matching on (C_*, Ω) is acyclic if and only if there exists a linear extension of $P(C_*, \Omega)$, such that in this extension $u(a)$ follows directly after a, for all $a \in \mathcal{D}(\Omega)$.*

This extension can always be chosen so that, restricted to $\mathcal{D}(\Omega) \cup \mathcal{C}(\Omega)$, it does not decrease the rank.

Proof. If such an extension L exists, then following a cycle (3) from left to right we always go down in the order L (more precisely, moving one position up is followed by moving at least two positions down), hence a contradiction.

Assume that the matching is acyclic, and define L inductively. Let Q denote the set of elements which are already ordered in L. We start with $Q=\emptyset$. Let W denote the set of the lowest rank elements in $P(C_*,\Omega)\setminus Q$. At each step we have one of the following cases:

Case 1. One of the elements c in W is critical. Then simply add c to the order L as the largest element, and proceed with $Q\cup\{c\}$.

Case 2. All elements in W are matched. The covering graph induced by $W\cup u(W)$ is acyclic, hence the total number of edges is at most $2|W|-1$. It follows that there exists $a\in W$, such that $P(C_*,\Omega)_{<u(a)}\setminus Q=\{a\}$. Hence, we can add elements a and $u(a)$ on top of L and proceed with $Q\cup\{a,u(a)\}$. □

Definition 1.4. Let (C_*,Ω) be a free chain complex with a basis, and let $\mathcal{M}$ be an acyclic matching. The *Morse complex* $\cdots \xrightarrow{\partial^{\mathcal{M}}_{n+2}} C^{\mathcal{M}}_{n+1} \xrightarrow{\partial^{\mathcal{M}}_{n+1}} C^{\mathcal{M}}_{n} \xrightarrow{\partial^{\mathcal{M}}_{n}} C^{\mathcal{M}}_{n-1} \xrightarrow{\partial^{\mathcal{M}}_{n-1}} \cdots$ is defined as follows. The $\mathcal{R}$-module $C^{\mathcal{M}}_n$ is freely generated by the elements of $\mathcal{C}_n(\Omega)$. The boundary operator is defined by $\partial^{\mathcal{M}}_n(s)=\sum_p w(p)\cdot p_\bullet$, for $s\in\mathcal{C}_n(\Omega)$, where the sum is taken over all alternating paths p satisfying $p^\bullet=s$.

Again, if the indexing is clear, we simply write $\partial^{\mathcal{M}}$ instead of $\partial^{\mathcal{M}}_n$.

Given a free chain complex with a basis (C_*,Ω), we can choose a different basis $\widetilde{\Omega}$ by replacing each $a\in\mathcal{D}_n(\Omega)$ by $\tilde a=w(u(a)\succ a)\cdot a$. Since

$$\mathfrak{K}_{\widetilde{\Omega}}(x,\tilde a)=\mathfrak{K}_{\Omega}(x,a)/w\bigl(u(a)\succ a\bigr),\tag{4}$$

for any $x\in\Omega_n$, we see that the weights of those alternating paths, which do not begin or end with an element from $\mathcal{D}_n(\Omega)$, remain unaltered, as the quotient $w(x\succ z)/w(y\succ z)$ stays constant as long as $x,y\neq a$. In particular, the Morse complex will not change. On the other hand, by (4), $w_{\widetilde{\Omega}}(u(a)\succ a)=1$, for all $a\in\mathcal{D}(\widetilde{\Omega})$, so the total weight of the alternating path in (2) will simply become

$$w_{\widetilde{\Omega}}(p)=(-1)^n w_{\widetilde{\Omega}}\bigl(s\succ d(b_1)\bigr)\cdot w_{\widetilde{\Omega}}\bigl(b_1\succ d(b_2)\bigr)\cdot\cdots\cdot w_{\widetilde{\Omega}}(b_n\succ t).$$

Because of these observations, we may always replace any given basis of C_* with the basis $\widetilde{\Omega}$ satisfying $w_{\widetilde{\Omega}}(u(a)\succ a)=1$, for all $a\in\mathcal{D}(\widetilde{\Omega})$.

2. The main theorem

The chain complex $\cdots\longrightarrow 0\longrightarrow\mathcal{R}\xrightarrow{\text{id}}\mathcal{R}\longrightarrow 0\longrightarrow\cdots$ where the only nontrivial modules are in the dimensions d and $d-1$, is called an *atom chain complex*, and is denoted by $\mathtt{Atom}(d)$.

The main result brings to light a certain structure in C_*. Namely, by choosing a different basis, we will represent C_* as a direct sum of two chain complexes, one of which is a direct sum of atom chain complexes, in particular acyclic, and the other one is isomorphic to $C^{\mathcal{M}}_*$. For convenience, the choice of basis will be performed in several steps, one step for each matched pair of the basis elements.

Theorem 2.1. *Assume that we have a free chain complex with a basis (C_*,Ω), and an acyclic matching $\mathcal{M}$. Then C_* decomposes as a direct sum of chain complexes $C^{\mathcal{M}}_*\oplus T_*$, where $T_*\simeq\bigoplus_{(a,b)\in\mathcal{M}}\mathtt{Atom}(\dim b)$.*

Proof. To start with, let us choose a linear extension L of the partially ordered set $P(C_*,\Omega)$ satisfying the conditions of the Proposition 1.3, and let $<_L$ denote the corresponding total order.

Assume first that C_* is bounded; without loss of generality, we can assume that $C_i=0$ for $i<0$, and $i>N$. Let $m=|M|$ denote the size of the matching, and let $l=|\Omega|-2m$ denote the number of critical cells.

We shall now inductively construct a sequence of bases $\Omega^0, \Omega^1, \ldots, \Omega^m$ of C_*. Furthermore, each basis will be divided into three parts: $\mathcal{C}(\Omega^k) = \{c_1^k, \ldots, c_l^k\}$, $\mathcal{D}(\Omega^k) = \{a_1^k, \ldots, a_m^k\}$, and $\mathcal{U}(\Omega^k) = \{b_1^k, \ldots, b_m^k\}$, such that $a_i^k = d(b_i^k)$, for all $i \in [m]$.

We start with $\Omega^0 = \Omega$ and the initial condition $b_i^0 <_L b_{i+1}^0$, for all $i \in [m-1]$. Since the lower index of $\mathfrak{K}_-(-,-)$ and $w_-(- \succ -)$ will be clear from the arguments, we shall omit it to make the formulae more compact.

When constructing the bases, we shall simultaneously prove by induction the following statements:

(i) $C_* = C_*[k] \oplus \mathcal{A}_1^k \oplus \cdots \oplus \mathcal{A}_k^k$, where $C_*[k]$ is the subcomplex of C_* generated by $\Omega^k \setminus \{a_1^k, \ldots, a_k^k, b_1^k, \ldots, b_k^k\}$, and $\mathcal{A}_i^k$ is isomorphic to $\mathtt{Atom}(\dim b_i^k)$, for $i \in [k]$;
(ii) for every $x^k \in \mathcal{U}(\Omega^k) \cup \mathcal{C}(\Omega^k)$, $y \in \mathcal{C}(\Omega^k)$, we have $w(x^k \succ y^k) = \sum_p w(p)$, where the sum is restricted to those alternating paths from x^0 to y^0 which only use the pairs (a_i^0, b_i^0), for $i \in [k]$.

Clearly, all of the statements are true for $k = 0$. Assume $k \geqslant 1$.

Transformation of the basis Ω^{k-1} into the basis Ω^k: set $a_k^k := \partial b_k^{k-1}$, $b_k^k := b_k^{k-1}$, and $x^k := x^{k-1} - w(x^{k-1} \succ a_k^{k-1}) \cdot b_k^{k-1}$, for all $x^{k-1} \in \Omega^{k-1}$, $x \neq a_k, b_k$.

First, we see that Ω^k is a basis. Indeed, assume $b_k^{k-1} \in C_n$. For $i \neq n, n-1$, we have $\Omega_i^k = \Omega_i^{k-1}$, hence, by induction, it is a basis. Ω_{n-1}^k is obtained from Ω_{n-1}^{k-1} by adding a linear combination of other basis elements to the basis element a_k^{k-1}, hence Ω_{n-1}^k is again a basis. Finally, Ω_n^k is obtained from Ω_n^{k-1} by subtracting multiples of the basis element b_k^{k-1} from the other basis elements, hence it is also a basis.

Next, we investigate how the poset $P(C_*, \Omega^k)$ differs from $P(C_*, \Omega^{k-1})$. If $x \neq b_k$, we have $w(x^k \succ a_k^k) = \mathfrak{K}(\partial x^k, a_k^k) = \mathfrak{K}(\partial x^k, a_k^{k-1}) = \mathfrak{K}(\partial x^{k-1}, a_k^{k-1}) - w(x^{k-1} \succ a_k^{k-1}) \cdot \mathfrak{K}(\partial b_k^{k-1}, a_k^{k-1}) = 0$, where the second equality follows from the fact that Ω_{n-1}^k is obtained from Ω_{n-1}^{k-1} by adding a linear combination of other basis elements to the basis element a_k^{k-1}, and the last equality follows from $\mathfrak{K}(\partial b_k^{k-1}, a_k^{k-1}) = 1$.

Furthermore, since Ω_n^k is obtained from Ω_n^{k-1} by subtracting multiples of the basis element b_k^{k-1} from the other basis elements, we see that for $x \in \Omega_{n+1}^k$, $y \in \Omega_n^k$, $y \neq b_k$, we have $w(x^k \succ y^k) = w(x^{k-1} \succ y^{k-1})$. Additionally, since the differential of the chain complex squares to 0, we have $0 = \sum_{z^k \in \Omega_n^k} w(x^k \succ z^k) \cdot w(z^k \succ a_k^k) = w(x^k \succ b_k^k) \cdot w(b_k^k \succ a_k^k) = w(x^k \succ b_k^k)$, where the second equality follows from $w(z^k \succ a_k^k) = 0$, for $z \neq b_k$.

We can summarize our findings as follows: all weights in the poset $P(C_*, \Omega^k)$ are the same as in $P(C_*, \Omega^{k-1})$, with the following exceptions:

(1) $w(x^k \succ b_k^k) = 0$, and $w(b_k^k \succ x^k) = 0$, for $x \neq a_k$;
(2) $w(a_k^k \succ x^k) = 0$, and $w(x^k \succ a_k^k) = 0$, for $x \neq b_k$;
(3) $w(x^k \succ y^k) = w(x^{k-1} \succ y^{k-1}) - w(x^{k-1} \succ a_k^{k-1}) \cdot w(b_k^{k-1} \succ y^{k-1})$, for $x \in \Omega_n^k$, $y \in \Omega_{n-1}^k$, $x \neq b_k$, $y \neq a_k$.

In particular, the statement (i) is proved. Furthermore, the following fact $(*)$ can be seen by induction, using (1), (2), and (3): if $w(x^k \succ y^k) \neq w(x^{k-1} \succ y^{k-1})$, then $b_k^0 \geqslant_L y^0$. Indeed, either $y \in \{a_k, b_k\}$, or y is critical, or $y = a_{\tilde{k}}$, for $\tilde{k} > k$, such that $w(b_k^{k-1} \succ y^{k-1}) \neq 0$. In the first two cases $b_k^0 \geqslant_L y^0$ by the construction of L, and the last case is impossible by induction, and again, by the construction of L.

We have $w(b_j^k \succ a_j^k) = w(b_j^{k-1} \succ a_j^{k-1})$, for all j, k. Indeed, this is clear for $j = k$. The case $j < k$ follows by induction, and the case $j > k$ is a consequence of the fact $(*)$.

Next, we see that the partial matching $\mathcal{M}^k := \{(a_i^k, b_i^k) \mid i \in [m]\}$ is acyclic. For $j \leqslant k$, the poset elements b_j^k, a_j^k are incomparable with the rest, hence they cannot be a part of a cycle. For $i > k$, we have $w(b_j^k \succ a_i^k) = w(b_j^{k-1} \succ a_i^{k-1})$, by the fact $(*)$. Hence, by induction, no cycle can be formed by these elements either.

Finally, we trace the boundary operator. Let $x^k \in \mathcal{U}(\Omega^k) \cup \mathcal{C}(\Omega^k)$, $y \in \mathcal{C}(\Omega^k)$. For $x = b_k$ the statement is clear. If $x \neq b_k$, we have $w(x^k \succ y^k) = w(x^{k-1} \succ y^{k-1}) - w(x^{k-1} \succ a_k^{k-1}) w(b_k^{k-1} \succ y^{k-1})$. By induction, the first term is counting the contribution of all the alternating paths from x^0 to y^0 which do not use the edges $b_l^0 \succ a_l^0$, for $l \geqslant k$. The second term contains the additional contribution of the alternating paths from x^0 to y^0 which use the edge $b_k^0 \succ a_k^0$. Observe, that if this edge occurs then, by the construction of L, it must be the second edge of the path (counting from x^0), and, by the fact $(*)$, we have $w(x^{k-1} \succ a_k^{k-1}) = w(x^0 \succ a_k^0)$. This proves the statement (ii), and therefore concludes the proof of the finite case.

It is now easy to deal with the infinite case, since the basis stabilizes as we proceed through the dimensions, so we may take the union of the stable parts as the new basis for C_*. $\square$

Remark 2. Even if the chain complex C^* is infinite in both directions, one can still define the notion of the acyclic matching and of the Morse complex. Since each particular homology group is determined by a finite excerpt from C^*, we may still conclude that $H_*(C_*) = H_*(C_*^{\mathcal{M}})$.

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