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Topology

A bilipschitz version of Hardt's theorem

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Abstract

In this Note we give a sketch of the proof of a theorem which is a bilipschitz version of Hardt's theorem. Given a family definable in an o-minimal structure Hardt's theorem states the existence (for generic parameters) of a trivialization which is definable in the o-minimal structure. We show that, for a polynomially bounded o-minimal structure, there exists such an isotopy which is bilipschitz. The proof is inspired by Bochnak et al. [Géométrie Algébrique Réelle, Springer-Verlag, 1987]. and involves the construction of 'Lipschitz triangulations' which are defined in this Note. The complete proof of existence will appear later. ***To cite this article: G. Valette, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une version bilipschitzienne du théorème de Hardt. Dans cette note on donne les grandes lignes de la preuve d'un théorème qui constitue une version bilipschitzienne du théorème de Hardt. Étant donnée famille d'ensembles définissables dans une structure o-minimale le théorème de Hardt établit l'existence d'une trivialisation topologique (pour des paramètres génériques) définissable dans la structure. On démontre que l'isotopie peut être choisie bilipschitzienne pour les structures o-minimales polynomialement bornées. La preuve consiste à démontrer l'existence de « triangulations lipschitz » simultanées (cf. Bochnak et al. [Géométrie Algébrique Réelle, Springer-Verlag, 1987]). On en donne ici l'idée et la définition ; la preuve détaillée de l'existence sera publié plus tard. ***Pour citer cet article : G. Valette, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Le théorème de Hardt est apparu dans [5] dans le cadre semi-algébrique. Il a ensuite été généralisé aux structures o-minimales. Ce théorème établit qu'une famille définissable dans une structure o-minimale est génériquement définissablement topologiquement triviale. On présente dans cette note une version bilipschitzienne de ce théorème

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pour une structure o-minimale polynomialement bornée (Théorème 1.2), c'est-à-dire que l'on construit génériquement des trivialisations définissables et bilipschitziennes pour de telles familles.

La preuve complète de ce théorème que l'on peut trouver dans [12] se rapproche de la preuve donnée dans [2] (du cas topologique) construisant la trivialisation à l'aide d'une triangulation des fibres génériques. Les triangulations lipschitz introduites à cet effet constituent un objet très naturel. En fait, en donnant une telle démonstration du Théorème 1.2 notre objectif est aussi d'introduire un objet combinatoire, comme peuvent être les triangulations pour l'étude topologique, renfermant la totalité du type lipschitzien. L'auteur pense que cet objet pourrait avoir d'autres applications notamment dans la construction d'invariants métriques.

La définition sera donnée dans la Section 3. Tentons ici d'en expliquer l'idée. Considérons le cas d'un cusp dans $\mathbb{R}^2$ d'équation $y^2 = x^3$. Si l'on veut trianguler cet ensemble il est impossible d'espérer que l'homéomorphisme h de triangulation soit bilipschitzienne. La seule chose que l'on puisse faire est de dire que h est un homéomorphisme et que, à travers h, les différences entre les abscisses sont préservées (au produit par une constante prés) et les différences entre les ordonnées sont multipliées par une puissance de la distance à l'origine (au produit par une constante prés aussi). Une construction similaire a été faite par Birbrair dans [1] dans le cas des courbes semi-algébriques. On la généralise ici au cas de la dimension quelconque. La triangulation lipschitz h d'un ensemble A sera alors un homémorphisme d'un complexe simplicial dans R^n telle que A soit réunion d'images de simplexes ouverts, « contractant » les simplexes suivant certaines directions. Les directions de contraction sont exprimées (sur chaque simplexe) dans des bases définies par des applications linéaires par morceaux et les contractions opèrent par produit par des fonctions qui sont des sommes finies de fonctions de la forme

$$d(q;\sigma_1)^{\alpha_1}\cdots d(q;\sigma_k)^{\alpha_k}$$

où $\sigma_1,\ldots,\sigma_k$ sont des simplexes, d la distance dans $\mathbb{R}^n$ et $\alpha_1,\ldots,\alpha_k$ des réels (éventuellement négatifs).

Ceci ramène l'étude du type lipschitzien à celle du type topologique jointe à celle du contact. C'est le théorème de préparation pour les fonctions définissables qui permet d'exprimer le contact comme un produit de distance [13,10,7]. Ce théorème avait d'ailleurs été introduit par Parusiński dans [9] pour prouver l'existence des stratifications lipschitziennes pour les ensembles sous-analytiques.

La preuve complète apparaîtra ultérieurement dans [12]. Une des difficultés est de trouver une projection régulière pour l'ensemble considéré, c'est à dire une projection linéaire sur un hyperplan dont l'angle formé par sa direction et le tangent à l'ensemble en un point régulier est borné inférieurement. Du moins on montre que ceci peut être obtenu à un homéomorphisme bilipschitzien prés. Pour cela on introduit la notion de « système régulier d'hypersurfaces » qui est une sorte de décompostion en ensembles L-réguliers [6,11] « orientée » et l'on démontre l'existence d'une telle famille compatible avec un ensemble définissable.

Une fois la projection régulière obtenue on construit la triangulation par récurrence sur la dimension de l'espace ambiant comme dans le cas topologique. Par un argument classique, la triangulation des fibres génériques [2,4] nous donne des triangulations simultanées qui induisent des trivialisations. La compacité de l'espace de Stone nous donne le recouvrement fini cherché.

La définissabilité de l'isotopie procure un meilleur contrôle des constantes lipschitz à l'approche des lieux de bifurcations. Ce théorème généralise la finitude des types lipschitziens au cas d'une famille définissable dans une structure o-minimale polynomialement bornée (l'existence des stratifications lipschitzienne n'a pas encore été prouvée au delà du cas sous-analytique) ainsi que pour les familles d'ensembles non-bornés (pas d'hypothèse de propreté).

1. Introduction and main result

Hardt's theorem appeared in [5] (for semi-algebraic sets). It states that a definable family of sets is generically definably topologically trivial. Here we claim that the trivialization can be chosen to be bilipschitz when the o-minimal structure is polynomially bounded. Generic bilispchitz triviality of subanalytic sets is known since the works of Mostowski and Parusiński but their isotopies are obtained by integration of vector fields and are not necessary subanalytic. Our proof is closer to the proof given in [2] or [3] in the topological case. We introduce a concept of Lipschitz triangulation to deduce the theorem from the triangulation of the generic fibers. So all we have to check is the existence of such triangulations. We believe that they constitute an interesting combinatorial tool to investigate metric properties of definable sets.

Throughout this note we fix a polynomially bounded o-minimal structure over $\mathbb{R}$. *The word definable will mean definable in this o-minimal structure.* Let A be a definable subset of $\mathbb{R}^n \times \mathbb{R}^p$. We will consider such a subset as a family of definable subsets of $\mathbb{R}^n$ parametrized by $\mathbb{R}^p$. For $U \subseteq \mathbb{R}^p$ we will denote by A_U the subfamily $\{q = (x; t) \in \mathbb{R}^n \times \mathbb{R}^p \mid q \in A, t \in U\}$, and for $t \in \mathbb{R}^p$ we will denote by A_t, the fiber of A at t, namely $\{x \in \mathbb{R}^n \mid q = (x; t) \in A\}$.

Definition 1.1. Let A be a definable subset of $\mathbb{R}^n \times \mathbb{R}^p$. We will say that A is *definably bilipschitz trivial along* $U \subseteq \mathbb{R}^p$ if there exists a definable homeomorphism $h : A_{t_0} \times U \to A_U$ of the form $h(x; t) = (h_t(x); t)$ together with a definable continuous function $C : U \to \mathbb{R}$ satisfying for any elements x and x' in A_{t_0}, and $t \in U$:

$$\left|h_t(x) - h_t(x')\right| \leqslant C(t) \cdot |x - x'| \tag{1}$$

and for any $(x; x') \in A_t \times A_t$, $t \in U$:

$$\left|h_t^{-1}(x) - h_t^{-1}(x')\right| \leqslant C(t) \cdot |x - x'|. \tag{2}$$

In this Note we sketch the proof of the following theorem:

Theorem 1.2. *Let A be a definable subset of $\mathbb{R}^n \times \mathbb{R}^p$. Then there exists a definable partition of $\mathbb{R}^p$, such that the family A is definably bilipschitz trivial along each element of this partition.*

Remark 1. (a) Note that the generic bilipschitz triviality was not known over an arbitrary polynomially bounded o-minimal structure, even by integration of vector fields. In Definition 1.1, the local Lipschitz constant is a function of the parameters. As it is a continuous function we have bilipschitz triviality over any closed bounded subset of each element of the partition. In the case of isotopies given by vector fields [8,9] constants of the flow tend to infinity as the exponential of the inverse of the distance to the boundary of the strata. In Theorem 1.2, the local Lipschitz constant obtained is better since it is a definable function which by Łojasiewicz's inequality can be bounded by a power of the inverse of the distance to the boundary of the strata.

(b) Note that, as in Hardt's theorem, in the above statement there is no assumption of properness. Thus, also at infinity, we obtain the finiteness of Lipschitzian types of the fibers. Note that Theorem 1.2 tells nothing about Lipschitzianity with respect to parameters. This could be obtained only in the bounded case [12].

2. Seeking regular projections

The first step in the proof of Theorem 1.2 is to find a regular projection (cf. Definition 2.1). Indeed we prove that we can find it for a definable set up to a definable bilipschitz homeomorphism. We first construct a kind of L-regular decomposition with an additional assumption ((ii) of Definition 2.3, see [11,6]).

Given $X \subseteq \bigcup_{k=1}^{n} \mathbb{G}_{k,n}$ we will denote by $\Lambda(X)$ the subspace of S^{n-1} constituted by all the unit vectors included in an element of X. Given a definable set $A \subseteq \mathbb{R}^n$ we define $\tau(A) \subseteq \bigcup_{k=1}^{n} \mathbb{G}_{k,n}$ as $\tau(A) = \mathrm{cl}\{T_x A \mid x \in A_{\mathrm{reg}}\}$.

Let $\lambda \in S^{n-1}$. We denote by $\pi_\lambda : \mathbb{R}^n \to N_\lambda$ the orthogonal projection onto the normal space of the vector λ, and by q_λ the coordinate of q along λ.

Let $A \subseteq \mathbb{R}^n$ be a definable set, $A' = \pi_\lambda(A)$ and $\xi : A' \to \mathbb{R}$ a definable function. The set A is said to be the graph of the function ξ for λ if: $A = \{q \in \mathbb{R}^n \mid q_\lambda = \xi(\pi_\lambda(q)), \pi_\lambda(q) \in A'\}$. The distance in $\mathbb{R}^n$ will be denoted by d.

Definition 2.1. Let A be a definable set of $\mathbb{R}^n$. An element λ of S^{n-1} is said to be *regular* for A if:

$$d\big(\lambda; \Lambda\big(\tau(A)\big)\big) \geqslant \alpha$$

with $\alpha \in \mathbb{R}^+$. A subset of S^{n-1} is said to be regular for A if all its elements are regular.

Proposition 2.2. *Let B be a connected subset of S^{n-1}, $\lambda_0 \in B$, and let $\xi : N_{\lambda_0} \to \mathbb{R}$ be a definable function. Let H be the graph of ξ for λ_0. Suppose that B is regular for H. Then, for any $\lambda \in B$ the set H is the graph of a function $\xi^\lambda : N_\lambda \to \mathbb{R}$. Moreover the set "under the graph", namely: $E(H; \lambda) = \{q \in \mathbb{R}^n \mid q_\lambda \leqslant \xi^\lambda(\pi_\lambda(q))\}$ is independent of $\lambda \in B$.*

This proposition says that the connected component of the ambient space which is under the graph with respect to a line λ depends only on the connected component of the complement of the tangent vectors to the graph in S^{n-1} where the line λ is chosen.

Now we come to our concept of regular family of hypersurfaces.

Definition 2.3. A *regular family of hypersurfaces* of $\mathbb{R}^n$ is a family $H = (H_k; \lambda_k)_{1 \leqslant k \leqslant b}$ of definable subsets of $\mathbb{R}^n$ together with elements λ_k of S^{n-1}, where $b \in \mathbb{N}$, such that the following properties hold for each $k < b$:

(i) The sets H_k and H_{k+1} are respectively the graphs for λ_k of two global Lipschitzian functions ξ_k and ξ'_k such that $\xi_k \leqslant \xi'_k$.
(ii) We have: $E(H_{k+1}; \lambda_k) = E(H_{k+1}; \lambda_{k+1})$.

Let A be a definable subset of R^n of empty interior. We say that the family H is *compatible* with A, if $A \subseteq \bigcup_{k=1}^{b} H_k$. An *extension* of H is a regular family compatible with the sets $\bigcup_{k=1}^{b} H_k$.

Note that by Proposition 2.2 the condition (ii) is implied by the following condition:

(iii) Let B_k be the connected component of $S^{n-1} \setminus \Lambda(\tau(H_k))$ containing λ_k, then $\lambda_k \in B_{k+1}$.

The idea is to change the direction and stay in the same connected component of the set of regular lines of the previous step.

Proposition 2.4. *For each definable set A of $\mathbb{R}^n$, of empty interior, there exists a regular family of hypersurfaces of $\mathbb{R}^n$ compatible with A.*

The following proposition is a consequence of the existence of regular systems of hypersurfaces. We give the proof to make understood to the reader how this tool should be used and why the previous proposition is useful in the proof of Theorem 1.2.

Proposition 2.5. *Let A be a definable subset of $\mathbb{R}^n$ of empty interior. Then there exists a definable bilipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(A)$ has a regular projection.*

Proof. By the above proposition there exists a regular system of hypersurfaces $(H_k; \lambda_k)_{1 \leqslant k \leqslant b}$ compatible with A. We define h on $E(H_k; \lambda_k)$, by induction on k, in such a way that $h(E(H_k; \lambda_k)) = E(F_k; e_n)$ (and so $h(H_k) = F_k$) where F_k is the graph of a function η_k for e_n, the last vector of the canonical basis of $\mathbb{R}^n$.

For $k = 1$ choose an orthonormal basis of N_{λ_1} and set $h(q) = (x_{\lambda_1}(q); q_{\lambda_1})$ where $x_{\lambda_1}(q)$ are the coordinates of $\pi_{\lambda_1}(q)$ in this basis. Let $k \geqslant 1$. By (i) the sets H_k and H_{k+1} are the graphs for λ_k of two Lipschitzian functions ξ_k and ξ'_k. For $q \in E(H_{k+1}; \lambda_k) \setminus E(H_k; \lambda_k)$ define $h(q) = h(\pi_{\lambda_k}(q); \xi_k(\pi_{\lambda_k}(q))) + (q_{\lambda_k}(q) - \xi_k(\pi_{\lambda_k}(q)))e_n$. By (ii) we have $E(H_{k+1};\lambda_{k+1}) = E(H_{k+1};\lambda_k)$, so we have extended h to $E(H_{k+1};\lambda_{k+1})$. Since ξ_k is Lipschitzian h is a bilipschitz homeomorphism. Note also that the image is $E(F_{k+1};e_n)$ where F_{k+1} is the graph of the Lipschitzian function $\eta_{k+1}(\pi_{e_n}(q)) = \eta_k(\pi_{e_n}(q)) + (\xi_k' - \xi_k) \circ \pi_{\lambda_k} \circ h^{-1}(q;\eta_k \circ \pi_{e_n}(q))$. This gives h over $E(H_b;\lambda_b)$. To extend h to the whole of $\mathbb{R}^n$ we proceed as in the case $k = 1$. □

3. Lipschitz triangulations

The proof of Theorem 1.2 is inspired by the proof given in [2] (for the topological case) involving the real spectrum. The reader can refer to [2] or [3] for notations and definitions.

Let us describe what is a Lipschitz triangulation. In fact this cannot be a bilipschitz homeomorphism. We shall require that over each simplex the distances are preserved up to 'some contractions' explicitly described along the directions of specific coordinate systems of $\mathbb{R}^n$. These contractions are each defined by sums of products of powers of distance to faces. Namely, given a simplex σ, we will consider functions $\phi_{\sigma,i}$ over σ in a complex K which are each a finite sum of functions of type:

$$d(q;\sigma_1)^{\alpha_1} \cdots d(q;\sigma_k)^{\alpha_k} \tag{3}$$

where $\sigma_1,\ldots,\sigma_k$ are simplices of K and $\alpha_1,\ldots,\alpha_k$ are real numbers (possibly negative). A finite sum of such functions will be called a *standard simplicial function*. Indeed Definition 3.2 will involve standard simplicial functions on $\sigma \times \sigma$ that are sums of distances involving q or another point q'.

The reader can refer to [3] for basic definitions about triangulations. Given a point $q \in \mathbb{R}^n$ we will write $q_1,\ldots,q_n$ for the coordinates of q in the canonical basis and $\pi_i : \mathbb{R}^n \to \mathbb{R}^i$ for the canonical projection.

Although the polyhedra will be subsets of $\mathbb{R}^n$ we will not be able to express the contractions using the canonical coordinates of $\mathbb{R}^n$. The reason is that the contractions operating on distances also modify the angles. So we will express the directions of contractions with specific local coordinates over each simplex defined by piecewise linear functions (for enlightenment on this notion see the proof of Theorem 5.1.3 of [12]):

Definition 3.1. Let σ be a simplex of $\mathbb{R}^n$. A *tame system of coordinates on* σ is a family of functions $(q_{1,\sigma};\ldots;q_{n,\sigma})$ of the following form:

$$q_{i,\sigma} = \frac{|q_i - \theta_i(\pi_{i-1}(q))|}{|\theta_i(\pi_{i-1}(q)) - \theta_i'(\pi_{i-1}(q))|} \tag{4}$$

(and 0 whenever $\theta_i \circ \pi_{i-1}(q) = \theta_i' \circ \pi_{i-1}(q)$) where θ_i and θ_i' are piecewise linear functions on $\mathbb{R}^{i-1}$.

These coordinates are defined inductively during the construction of the triangulation which is built-up by induction on n (as for usual triangulations). Now:

Definition 3.2. A *Lipschitz triangulation of* $\mathbb{R}^n$ is the data of a finite simplicial complex K together with a homeomorphism $h : |L| \to \mathbb{R}^n$, where L is a union of open simplices of K, such that for every $\sigma \in L$ there exist $\varphi_{\sigma,1},\ldots,\varphi_{\sigma,k}$, standard simplicial functions over $\sigma \times \sigma$ satisfying for any q and q' in σ:

$$\left|h(q) - h(q')\right| \sim \sum_{i=1}^{n} \varphi_{\sigma,i}(q;q') \cdot |q_{i,\sigma} - q'_{i,\sigma}| \tag{5}$$

where $(q_{1,\sigma},\ldots,q_{n,\sigma})$ is a tame system of coordinates on σ. Let $A_1,\ldots,A_k$ be subsets of $\mathbb{R}^n$. A Lipschitz triangulation of $A_1,\ldots,A_k$ is a Lipschitz triangulation of $\mathbb{R}^n$ such that each $h^{-1}(A_i)$ is a union of open simplices. Here $\sim$ means that the quotient of both sides is bounded above and below on $\sigma \times \sigma$.

With this definition two definable subsets admitting the same simplicial complex as definable triangulation, with $\sim$ functions φ_σ and the same tame systems of coordinates are definably bilipschitz homeomorphic. So simultaneous Lipschitz triangulations of fibers of a family will provide definable trivializations.

Theorem 3.3. *Let $A_1, \ldots, A_k$ be definable subsets of $\mathbb{R}^n$. Then there exists a definable Lipschitz triangulation of $A_1, \ldots, A_k$.*

The proof is done by induction on n. The crucial points are Theorem 2.5 which states the existence of a good projection and the Preparation Theorem for definable functions [13] (introduced in [10] to prove existence of Lipschitz stratifications for subanalytic sets) which implies the following proposition:

Proposition 3.4. *Let $\xi : \mathbb{R}^n \to \mathbb{R}$ be a positive definable function. Then there exists a partition of $\mathbb{R}^n$ such that over each element of this partition the function ξ is $\sim$ to a product of powers of distances to definable subsets of $\mathbb{R}^n$.*

Proof of Theorem 1.2. Let $\alpha \in \widetilde{\mathbb{R}^p}$ (see [3,2]) and let $(h; K)$ be a Lipschitz triangulation of A_α (Theorem 3.3 works over $k(\alpha)$ also, see [12] for details). We may suppose that this triangulation has its vertices in $\mathbb{Q}^n$. By standard arguments (see [3]) we may find $U_\alpha \in \alpha$, a simplicial complex $K' \subseteq \mathbb{R}^n$ with $|K'|_{k(\alpha)} = |K|$, and a mapping $H : L' \times U_\alpha \to \mathbb{R}^n \times U_\alpha$, with L' union of some open simplices of K' such that each $(H_t; K')$ is a Lipschitz triangulation of A_t. Let $\psi_t = H_t \circ H_{t_0}^{-1}$ for some $t_0 \in U_\alpha$. Then, by definition of Lipschitz triangulations, each ψ_t is a bilipschitz homeomorphism. This implies that ψ_α is a bilipschitz homeomorphism. Then, there exists $U'_\alpha \in \alpha$ such that the mapping $\psi : A_{t_0} \times U'_\alpha \to A_{U'_\alpha}$ defined by $\psi(x; t) = (\psi_t(x); t)$ is a family of bilipschitz homeomorphisms. The sets $\widetilde{U'_\alpha}$ constitute an open covering of $\widetilde{\mathbb{R}^p}$, by compactness of this set we have the desired covering. □

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