



Partial Differential Equations

# Convergence to the equilibrium for the Pauli equation without detailed balance condition

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## Abstract

We prove that for  $\rho \in (0, 1)$ , the homogeneous Boltzmann–Pauli equation, without detailed balance condition on the cross-section, has a unique steady state of total charge  $\rho$ . Moreover, we show that the solutions to the Cauchy problem converge to this steady state, as  $t$  tends to infinity. *To cite this article: N. Ben Abdallah et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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## Résumé

**Convergence vers l'équilibre des solutions de l'équation de Pauli sans hypothèse d'équilibre en détail.** Nous montrons que pour tout  $\rho \in (0, 1)$ , il existe un unique état stationnaire de charge totale  $\rho$  pour l'équation de Boltzmann–Pauli homogène, sans hypothèse d'équilibre en détail sur la section efficace. Nous montrons ensuite la convergence en temps grand vers cet équilibre des solutions du problème de Cauchy. *Pour citer cet article : N. Ben Abdallah et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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## Version française abrégée

Nous étudions le problème de Cauchy associé à l'équation de Boltzmann–Pauli homogène. Celle-ci est un modèle simplifié décrivant la dynamique d'un gaz de particules (ions ou électrons) dans un semiconducteur. Nous

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dénotons  $f = f(t, k)$  la densité de particules de vecteur d'onde  $k \in B$  à l'instant  $t \geq 0$ , où  $B$  représente la première zone de Brillouin (le tore normalisé de  $\mathbb{R}^N$ ,  $N \geq 1$ ,  $|B| = 1$ ). L'équation de Boltzmann–Pauli homogène s'écrit :

$$\frac{\partial f}{\partial t} = \operatorname{div}_k(Ef) + Q(f) \quad \text{in } (0, +\infty) \times B, \quad f(0, k) = f^{\text{in}}(k) \quad \text{on } B. \quad (1)$$

Le champ électrique  $E$  satisfait  $E \in W^{1,1}(B)$ ,  $\operatorname{div}_k E = 0$  et l'opérateur de Boltzmann–Pauli  $Q(f)$ , est donné par :

$$Q(f)(k) = \int_B (\sigma(1-f)f' - \sigma'(1-f')f) dk', \quad (2)$$

où l'on a posé  $f = f(k)$ ,  $f' = f(k')$ ,  $\sigma = \sigma(k, k')$  et  $\sigma' = \sigma(k', k)$ . Nous insistons sur le fait que nous ne faisons pas l'hypothèse d'équilibre en détail.

L'Éq. (1) possède deux propriétés importantes. La première est que le nombre total de particules, ou charge totale,  $N(f) = \int_B f(k, t) dk$  est indépendant du temps. D'autre part l'Éq. (1) définit une contraction stricte dans les sous ensembles de fonctions de  $L^1(B)$  de charge totale donnée. Grâce à ces deux propriétés, à l'hypothèse (H1) de positivité ci-dessous et au théorème du point fixe de Tykhonov, nous démontrons que pour tout  $\rho \in (0, 1)$  il existe une unique solution stationnaire  $F_\rho$  dans l'ensemble  $X_\rho$  défini ci-dessous (cf. (4)).

Nous considérons ensuite le problème d'évolution. Nous montrons tout d'abord, toujours sous l'hypothèse (H1), que pour toute donnée initiale  $f_{\text{in}} \in X_\rho$  il existe une unique solution  $f \in C([0, \infty), L^1(B))$  qui conserve la charge totale. Nous démontrons ensuite, sous une hypothèse plus restrictive (H2), que  $f(t)$  converge vers  $F_\rho$  lorsque  $t$  tend vers  $+\infty$ . Nous utilisons pour ceci la propriété de contraction stricte à charge totale fixée et le principe d'invariance de Lasalle. Finalement, sous une hypothèse (H3) encore plus forte, nous démontrons la convergence avec taux exponentiel.

## 1. Introduction

We study the Cauchy problem for the spatially homogeneous Boltzmann–Pauli equation considered as a simplified model for the dynamics of a cloud of particles (electrons or ions) in a semiconductor device. We denote  $f = f(t, k) \geq 0$  the density of particles with wave vector  $k \in B$  at time  $t \geq 0$ , being  $B$  the first Brillouin zone (the normalized torus of  $\mathbb{R}^N$ ,  $N \geq 1$ , so that  $|B| = 1$ ). The Boltzmann–Pauli equation reads:

$$\frac{\partial f}{\partial t} = \operatorname{div}_k(Ef) + Q(f) \quad \text{in } (0, +\infty) \times B, \quad f(0, k) = f^{\text{in}}(k) \quad \text{on } B, \quad (1)$$

where the electric field  $E \in W^{1,1}(B)$  satisfies  $\operatorname{div}_k E = 0$  and the Boltzmann–Pauli operator  $Q(f)$  is:

$$Q(f)(k) = \int_B (\sigma(1-f)f' - \sigma'(1-f')f) dk', \quad (2)$$

and where we have set  $f = f(k)$ ,  $f' = f(k')$ ,  $\sigma = \sigma(k, k')$  and  $\sigma' = \sigma(k', k)$ .

We emphasize that we do not make the so-called detailed balance condition hypothesis which usually reads as follows:  $\sigma M = \sigma' M'$  on  $B^2$  where  $M(k) = \exp(-\mathcal{E}(k))$  is the Maxwellian associated to the energy  $\mathcal{E}$ . We introduce instead the following assumptions on the cross section  $0 \leq \sigma \in L^1(B^2)$ , defining  $\underline{\sigma} = \min(\sigma, \sigma')$ ,

$$\sigma > 0 \quad \text{a.e.}, \quad (H1)$$

$$\sigma_1 := \int_B \operatorname{infess}_k \underline{\sigma} dk' > 0, \quad \text{and} \quad \sigma \in W^{1,1}(B^2), \quad (H2)$$

$$\operatorname{infess}_{k'} \sigma + \operatorname{infess}_{k'} \sigma' \neq 0 \quad \text{on } B. \quad (H3)$$

For any integrable density  $f$  the total number of particles, or total charge, is given by

$$N(f) = \int_B f(t, k) dk. \tag{3}$$

An important property of Eq. (1) is that, at least formally, the total number of particles  $N(f)$  is constant in time. We introduce then the sets:

$$X := \{f \in L^\infty(B), 0 \leq f \leq 1\} \quad \text{and} \quad X_\rho := \{f \in X, N(f) = \rho\}. \tag{4}$$

## 2. The main result

The main result of this Note is the following.

**Theorem 2.1.** 1. Assume (H1). For all initial data  $f_{\text{in}} \in X_\rho$ ,  $\rho \in (0, 1)$ , there exists a unique solution  $f \in C([0, \infty); L^1)$  of (1), (2). Moreover it preserves the total charge, i.e.  $f(t) \in X_\rho$  for all  $t \geq 0$ .

2. Assume (H1). For any  $\rho \in (0, 1)$  there exists a unique stationary solution  $F_\rho \in X_\rho$ . Moreover, if  $r < \rho$  then,  $F_r < F_\rho$  almost everywhere in  $B$ . In particular,  $0 < F_\rho < 1$ .

3. Assume (H1), (H2). For any initial data  $f_{\text{in}} \in X_\rho$ , the associated solution satisfies

$$H_\rho(f(t, \cdot)) := \int_B |f(t, k) - F_\rho(k)| dk \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

4. Assume (H1), (H3). For all initial data  $f_{\text{in}} \in X_\rho$ , the solution  $f$  converges exponentially fast towards  $F_\rho$  in  $L^1(B)$  as  $t \rightarrow +\infty$ :  $\exists \lambda_\rho > 0$

$$H_\rho(f(t, \cdot)) \leq H_\rho(f_{\text{in}}) e^{-\lambda_\rho t} \quad \forall t \geq 0.$$

**Remark 1.** The Pauli equation without detailed balance condition was first considered in [4]. The existence result of a stationary solution has been proved in [1] under the hypothesis of detailed balance and for a regular cross section  $\sigma$ . It was then adapted in [8,9] without the detailed balance and for a vanishing force  $E$ . The result presented here is an improvement of the above results under the general and natural condition on  $\sigma$  (condition (H1)). The proof uses the Tykhonov fixed point theorem, introduced in the field of kinetic equations in [7]. This method is developed in [6,11] to prove the existence of stationary states and self similar solutions for coagulation, fragmentation and inelastic collisions models.

On the other hand, we prove two convergence results for the solutions to the Cauchy problem towards the corresponding stationary solution. Our proof relies on an  $L^1$  contraction property in the spirit of [10,2].

The proof of Theorem 2.1 uses the following auxiliary result.

**Lemma 2.2.** Let us define

$$D_j(f, g) := \int_B (Q(f) - Q(g)) \xi'_j(f - g) dk, \quad j = 0, 1,$$

where  $\xi_0(s) = s_+$  (and then  $\xi'_0(s) = \mathbf{1}_{s>0}$ ) and  $\xi_1(s) = |s|$  (and then  $\xi'_1(s) = \mathbf{1}_{s>0} - \mathbf{1}_{s<0}$ ).

(i) Suppose (H1). For all  $f \in X_\rho$ ,  $g \in X_r$ ,  $r, \rho \in (0, 1)$ , there holds

$$D_j(f, g) \geq 0 \quad \text{and} \quad (D_j(f, g) = 0, \rho \geq r \implies f \geq g \text{ a.e.}).$$

(ii) Assume (H3). For all  $f, g \in X_\rho$  with  $0 < g < 1$  a.e. there exists  $\lambda_g > 0$  such that

$$D_1(f, g) \geq \lambda_g \|f - g\|_{L^1(B)}.$$

**Proof of Lemma 2.2.** Let us start with the point (i). We notice that

$$\begin{aligned} D_j(f, g) &= \int \int_{B \times B} \sigma[(1-f)(f' - g') - (f-g)g'](\xi'_j(f' - g') - \xi'_j(f-g)) \, dk \, dk' \\ &= \int \int_{B \times B} \sigma(1-f) \{ \xi_j(f' - g') - (f' - g') \xi'_j(f-g) \} \, dk \, dk' \\ &\quad + \int \int_{B \times B} \sigma g' \{ \xi_j(f-g) - (f-g) \xi'_j(f' - g') \} \, dk \, dk'. \end{aligned}$$

This implies  $D_j(f, g) \geq 0$ , since the functions under the integral signs in the right-hand side of the second equality are nonnegative. Since  $D_j(f, g) = D_j(g, f)$ , we may interchange the role played by  $f$  and  $g$ , and we get

$$\begin{aligned} D_j(f, g) &= \frac{1}{2} \int \int_{B \times B} [\sigma(1-f+1-g) + \sigma'(f+g)] \{ \xi_j(f' - g') - (f' - g') \xi'_j(f-g) \} \, dk \, dk' \\ &\geq \int \int_{B \times B} \underline{\sigma} \{ \xi_j(f' - g') - (f' - g') \xi'_j(f-g) \} \, dk \, dk'. \end{aligned} \quad (5)$$

When  $D_j(f, g) = 0$  we then deduce that  $\xi_j(f' - g') - (f' - g') \xi'_j(f-g) = 0$  a.e. on  $B \times B$ , so that  $\text{sign}(f-g)$  is a constant. The total charge condition  $f \in X_\rho, g \in X_r, \rho \geq r$ , allows to conclude that  $f \geq g$ .

In order to prove (ii) under Hypothesis (H3), we proceed as in [2] where the detailed balance case is treated. Indeed, since  $f$  and  $g$  have the same total charge, we deduce from (5):

$$\begin{aligned} D_1(f, g) &\geq \frac{1}{2} \int \int_{B \times B} \left[ \inf_{k'} \sigma(1-g) + \inf_{k'} \sigma' g \right] [|f' - g'| - (f' - g') \text{sign}(f-g)] \, dk \, dk' \\ &\geq \frac{1}{2} \int_B \left[ \inf_{k'} \sigma + \inf_{k'} \sigma' \right] \min(g, 1-g) \, dk \int_B |f' - g'| \, dk'. \quad \square \end{aligned}$$

**Proof of Theorem 2.1, 1st point.** Uniqueness is an immediate consequence of Lemma 2.2 since, if  $f$  and  $g$  are two solutions of solutions of (1), (2) with same total charge:

$$\frac{d}{dt} \int_B |f - g| \, dk = -D_1(f, g) \leq 0. \quad (6)$$

To prove the existence of solutions, we introduce a sequence of regularized cross sections  $\sigma_n$ , satisfying for instance (H1)–(H3), and such that  $\sigma_n \nearrow \sigma$  a.e. in  $B^2$ . Then, for all initial data  $f_{\text{in}} \in X_\rho$ , there exists a solution  $f_n \in C([0, \infty); L^1) \cap L^\infty(0, \infty; X_\rho)$  to the corresponding Boltzmann–Pauli equation (see e.g. [12,13]). To pass to the limit in the regularized problem, we make use of the following stability result, whose proof is classical.

**Lemma 2.3 (Stability Principle).** Let  $(\sigma_n)$  be a sequence of cross-sections such that  $\sigma_n \rightarrow \sigma$  in  $L^1$  and a.e. on  $B^2$ . Let  $(f_n)$  be a sequence of solutions to the Boltzmann–Pauli equation associated with the cross sections  $(\sigma_n)$  and uniformly bounded in  $L^\infty(0, \infty; X_\rho)$ . There exists  $f \in C([0, \infty); L^1) \cap L^\infty(0, \infty; X_\rho)$  and a subsequence  $(f_{n'})$  such that  $f_{n'} \rightharpoonup f$  in  $C([0, \infty); L^\infty\text{-weak})$  and  $f$  is a solution to the Boltzmann–Pauli equation with the cross section  $\sigma$ .

**Proof of Theorem 2.1, 2nd point.** By the stability principle above and the uniqueness of solutions, for any  $n$ , the map  $S_{2^{-n}} : X_\rho \rightarrow X_\rho$ ,  $f_{\text{in}} \mapsto f(2^{-n})$  is continuous and compact for the weak topology  $\sigma(L^\infty, L^1)$ . Then, by the Tykhonov’s fixed point theorem (see e.g. [5]), there exists at least one  $2^{-n}$  periodic solution  $g_n$  of the Boltzmann–Pauli equation:  $g_n \in X_\rho$  and  $S_{2^{-n}} g_n = g_n$ . Since the sequence  $(g_n)$  is bounded, by the stability principle,  $g_n(t) \rightharpoonup g(t)$  weakly  $\sigma(L^\infty, L^1) \forall t \geq 0$ , where  $g \in X_\rho$  is a solution satisfying  $S_t g = g$  for all dyadic time  $t \geq 0$ . We deduce  $S_t g = g$  for all  $t > 0$  and  $F_\rho := g \in X_\rho$  is a stationary solution.

The property  $0 \leq F_\rho \leq 1$  is satisfied by construction. By Lemma 2.2 the map  $\rho \rightarrow F_\rho$  is nondecreasing. Let consider  $\rho > r$  and define  $g := F_\rho - F_r \geq 0$ . We easily check that  $g$  satisfies

$$E \cdot \nabla_k g - \lambda g = -S$$

with  $\lambda(k) := \int_B [\sigma F'_\rho + \sigma'(1 - F'_r)] dk' \in L^1(B)$ ,  $S(k) := \int_B [\sigma(1 - F_r) + \sigma' F_\rho] g' dk'$ .

We deduce that

$$\forall \varepsilon > 0 \quad \int_B \lambda(\varepsilon - g)_+ dk = \int_B (\lambda \varepsilon - S) \mathbf{1}_{\varepsilon - g \geq 0} dk.$$

Passing to the limit  $\varepsilon \rightarrow 0$ , we obtain  $\int_B S \mathbf{1}_A dk = 0$ , where  $A := \{g = 0\}$ . Since  $S > 0$  a.e. we deduce that  $\text{mes}(A) = 0$  and then  $\rho \mapsto F_\rho$  is strictly increasing.

The following result is needed to prove the asymptotic convergence of the solution to the stationary state as  $t \rightarrow +\infty$ .

**Lemma 2.4.** Assume (H2). If  $f_{\text{in}} \in X_\rho$  is such that  $\partial_k f_{\text{in}} \in L^1$ , then,

$$\sup_{t > 0} \int_B |\partial_k f(t, k)| dk < \infty. \tag{7}$$

**Proof of Lemma 2.4.** Since

$$\partial_k Q(f) = \int_B ((\partial_1 \sigma)(1 - f)f' - (\partial_2 \sigma)'(1 - f')f) dk' - \int_B (\sigma f' + \sigma'(1 - f')) dk' \partial_k f,$$

and  $\sigma f' + \sigma'(1 - f') \geq \underline{\sigma} f' + \underline{\sigma}(1 - f') = \underline{\sigma}$ , we obtain thanks to (H2)

$$\frac{d}{dt} \|\partial_k f\|_{L^1} = \int_B \partial_k Q(f) \text{sign}(\partial_k f) dk \leq \|\partial_1 \sigma\|_{L^1} + \|\partial_2 \sigma\|_{L^1} - \sigma_1 \|\partial_k f\|_{L^1}.$$

The estimate (7) follows from this differential inequality.  $\square$

**Proof of Theorem 2.1, 3rd point.** Consider  $f_{\text{in}} \in X_\rho$  and suppose also for the moment that  $\partial_k f_{\text{in}} \in L^1$ . Let us show that  $H_\rho(f(t))$  is a strict Lyapunov functional on  $L^1(B)$ . Since  $\frac{d}{dt} H_\rho(f) = -D_1(f, F_\rho)$ , we deduce, by the uniqueness result, that the map  $t \rightarrow H_\rho(f(t))$  is strictly decreasing as long as  $f \neq F_\rho$ . Moreover, by Lemma 2.4,  $(f(t))_{t > 0}$  belongs to a compact subset of  $L^1(B)$ . Then, the Lasalle’s invariance principle (see e.g. [3]) implies that  $H_\rho(f(t)) \rightarrow 0$  when  $t \rightarrow +\infty$ . The result for a general initial data  $f_{\text{in}} \in X_\rho$  follows by density using Lemma 2.2.

**Proof of Theorem 2.1, 4th point.** It is an immediate consequence of Lemma 2.2 and (6).  $\square$

**Remark 2.** For the uniqueness in points (i) and (ii) of Theorem 2.1, condition (H1) may be weakened. For instance, these uniqueness results still hold assuming  $\sigma + \sigma' > 0$  a.e. on  $B \times B$  or assuming  $\underline{\sigma} > 0$  a.e. on  $A \times B$  for some  $A \subset B$  such that  $\text{mes}(A) > 0$ . As an example, we may choose  $\sigma = \mathbf{1}_{k, k'; k_N \geq k'_N}$  which satisfies the first condition

but not the second one nor (H1) and  $\sigma = \mathbf{1}_{A \times B} + \mathbf{1}_{B \times A}$  (for some  $A \subset B$  such that  $\text{meas}(A) > 0$ ) which satisfies the second condition but not the first one nor (H1). Let us also emphasize that, at least when  $E \equiv 0$ , the strict monotonicity of the map  $\rho \mapsto F_\rho$  also holds for the cross section  $\sigma := \mathbf{1}_{k, k'; k_N + \varepsilon \geq k'_N}$ ,  $\varepsilon > 0$ , which does not satisfy the assumption (H1).

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