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Functional graphs

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Abstract

A graph G is said to be a functional graph if there exist two mappings f and g from $V(G)$ into a set F such that xy is an edge in G whenever $f(x) = g(y)$ or $g(x) = f(y)$. Chvátal and Ebenegger proved that recognizing functional graphs is an NP-complete problem. Using the compactness theorem, we prove that if G is an infinite graph such that any finite subgraph of G is a functional graph, then G is a functional graph. We give an elementary proof of this fact in the infinite countable case. In the finite case, we prove that for n large enough, any graph of girth n containing at most $3n - 7$ vertices is a functional graph. It will be shown by an example that this bound is the best possible. ***To cite this article: A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Graphes fonctionnels. Un graphe G est dit graphe fonctionnel s'il existe deux applications f et g de $V(G)$ dans un ensemble F telles que xy est une arête de G si et seulement si $f(x) = g(y)$ ou $g(x) = f(y)$. Chvátal et Ebenegger ont prouvé que le problème de reconnaissance des graphes fonctionnels est NP-complet. En utilisant le théorème de compacité, nous prouvons que si G est un graphe infini tel que tout sous-graphe fini de G est fonctionnel, alors G est fonctionnel. Nous donnons une preuve élémentaire de ce fait dans le cas dénombrable. Dans le cas fini, nous prouvons que pour n suffisamment grand, tout graphe sans cycle d'ordre plus petit que n et contenant au plus $3n - 7$ sommets est un graphe fonctionnel. Il sera montré à l'aide d'un exemple que $3n - 7$ est la meilleure borne possible. ***Pour citer cet article : A. El Sahili, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

The graphs and digraphs considered here may be infinite. They have no loops or multiple edges. A digraph may have directed cycles of length two. When $a = (x, y)$ is an edge of D then x is said to be the tail of a, and y is

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the head of a. We write $t(a) = x$ and $h(a) = y$. Multigraphs and multidigraphs are obtained when multiple edges are allowed. We refer to [5] and [6] for a good knowledge of compactness theorem and basic definitions in Model Theory. If H is a subgraph of connected graph G, $G - H$ denotes the graph obtained from G by deleting first the edges of H and then the isolated vertices. By $G(D)$ we denote the underlying graph of a digraph D that is the graph obtained by ignoring the orientation of the edges in D. (A cycle of length two corresponds to a unique edge.)

Let D be a multidigraph. *The line digraph $L(D)$* of D is the digraph whose vertex set is the edge set of D, where (e, f) is an edge in $L(D)$ if $h(e) = t(f)$. Based on [3] we may easily remark that functional graphs are exactly the underlying graphs of line digraphs defined simply in terms of functions. In fact, let f and g be two mappings from $V(G)$ into a set F which satisfy: $xy \in E(G) \Leftrightarrow f(x) = g(y)$ or $g(x) = f(y)$ for every x and y in $V(G)$.

On $V(G)$ we define a multidigraph D by putting α edges from a to b where $\alpha = |g^{-1}(a) \cap f^{-1}(b)|$ for all $a, b \in V(G)$, and a digraph H such that $(x, y) \in E(H)$ if $f(x) = g(y)$.

We associate to each edge (a, b) in D a unique vertex x in $V(H)$ such that $g(x) = a$ and $f(x) = b$. Then (x, y) is an edge in H if $h(x) = t(y)$ (as edges in D). Thus $H = L(D)$. It may be easily remarked that $G = G(H)$.

Beineke in [1] characterizes line digraphs as follows:

Theorem 1.1. *A digraph H is a line digraph if and only if whenever a, b and c are any three edges in H such that $h(a) = h(b)$ and $t(b) = t(c)$, there exists an edge d in H such that $t(d) = t(a)$ and $h(d) = h(c)$.*

Chvátal and Ebenegger [2] prove that recognizing underlying graphs of line digraphs is an NP-complete problem.

By simply remarking that the construction of Chvátal and Ebenegger leads to a square free graph, we noted in [4] that Chvátal and Ebenegger's proof implies even more, namely that recognizing underlying graphs of a line digraphs of digraph in which each vertex has in-degree or out-degree at most one is an NP-complete problem.

Let G be an infinite graph such that any finite subgraph of G is a functional graph. It is normal to ask whether G is a functional graph or not. We treat this problem in the next section.

2. Infinite functional graphs

A direct application of the compactness theorem of model theory yields the following theorem:

Theorem 2.1. *Let G be an infinite graph such that any finite subgraph of G is a functional graph. Then G is a functional graph.*

This theorem can also be established using ultrafilter's axiom, but without using the axiom of choice required by the compactness theorem. In the infinite countable case, we may establish an elementary proof based on induction. This proof gives a little bit more. In fact the theorem may be improved by replacing subgraphs by only induced subgraphs. It may be shown by examples that a subgraph of a functional graph may be a non functional graph, while the induced subgraph of a functional graph is always a functional graph.

3. Finite functional graphs

As a consequence of Beineke theorem, a functional graph cannot contain the complete graph K_4 as subgraph. Hence K_4 is a simple example of finite non functional graph such that any proper subgraph (distinct from the graph itself) is functional. Other examples without complete subgraph can be constructed. More precisely, we define for all $n \geqslant 5$ a non functional graph G of girth n such that any proper subgraph is a functional graph. We start by the following trivial fact.

Proposition 3.1. *A graph G containing at most one cycle is a functional graph.*

Definition 3.2. A graph G with $\Delta(G) \leqslant 3$ is called small if it is defined by three paths having the same ends and no other intersection, some vertices of the paths are joined to vertices of degree 1 in such a way that two vertices on the same path having degree two are adjacent.

If G is a graph, we denote by G_2 the subgraph of G induced by the vertices of degree two and by G_3 the subgraph induced by those of degree at least three. A 2-*vertex* of G is a vertex whose neighbors are all of degree at most two. If $A \subseteq V(G)$, we denote by $V_2(A)$ the set of the vertices having distance at most 2 from some vertices in A. The following lemma is a simple remark on line digraphs:

Lemma 3.3. *Let G be a functional graph of girth at least 5, and let L be a line digraph such that $G = G(L)$. If S is a connected component of G_3 containing exactly one cycle then $d_L^-(v) = d_S^-(v) = 1$ for all $v \in S$ or $d_L^+(v) = d_S^+(v) = 1$ for all $v \in S$.*

Proposition 3.4. *Let G be a small graph of girth at least 5 and having exactly two vertices of degree two. If these vertices belong to the same path, then G is not a functional graph and $G - v$ is functional for all vertex v in G.*

Corollary 3.5. *Let G be a graph obtained from a small graph G_s by adding to it a set of vertices (possibly empty) of degree exactly one joined to vertices of odd degree in G_s. If G_s contains at least 3 vertices of degree two, then G is a functional graph. We call it a* good small graph.

Inspired by the above proposition, we asked about a lower bound of the orders of non functional graphs of girth n. We are led to prove the following result:

Theorem 3.6. *If $n \geqslant 30$, then any graph G of girth n such that $v(G) \leqslant 3n - 7$ is a functional graph. This bound is the best possible.*

Definition 3.7. A chord of a cycle C is a path intersecting C only at its ends. An l-chord of C is a chord of length at most l. s chords of a cycle are said to be free if they may only intersect at their ends. Two free chords are said to be parallel when C may be the outer face of the planar graph formed by C together with these two chords.

To prove Theorem 3.6, we need some lemmas:

Lemma 3.8. *Let G be a bipartite planar graph with no cut-vertex. Suppose that $V = X \cup Y$ where X and Y are two stables such that $V(G_3) \subseteq X$. Then $|Y| - |X| = f - 2$, where f is the number of faces of G.*

Corollary 3.9. *Let G be a planar graph with f faces and no cut-vertex. Let M be a multigraph obtained by adding to G a stable set S of s new vertices such that each vertex of S is joined to exactly one vertex of G by a double edge. If $V(M) = X \cup Y$ where X and Y are two stables such that $V(M_3) \subseteq X$, then $|Y| - |X| = f(M) - 2$. ($f(M)$ is the number of faces of M.)*

Lemma 3.10. *Let G be a planar graph of girth n. Then $v(G) \geqslant \frac{nf}{2} - f + 2$, where f is the number of faces of G.*

Lemma 3.11. *Let G be a graph of girth $n \geqslant 30$ such that $v(G) \leqslant 3n - 7$ and suppose that G has no cycles with 4-chord. If G_p is a planar subgraph of G with 4 faces and no cut-vertex, the induced subgraphs of G by $V(G_3) \cap V(G_p)$ and $V(G_2) \cap V(G_p)$ are denoted by T and T' respectively. Then either T' contains a connected component with at least three vertices or T contains a connected component C such that $|V_2(C) - V(G_p)| \leqslant 2$ and C contains no vertex v such that $d_{G_p}(v) \geqslant 3$.*

Remark 1. Using the same arguments, we may get the same sequences even if G_p has less than 4 faces but at least $2n-5$ vertices.

Lemma 3.12. *Let G be a graph of girth $n \geqslant 30$ such that $v(G) \leqslant 3n-7$ and suppose that G has no cycles with* 4*-chord. Then one of the following statements holds*:

1. *G has at most one cycle.*
2. *G is a good small graph.*
3. *G contains a* 2*-vertex.*
4. *There is a subtree T of G containing a single pair of non adjacent vertices of G_2 which are joined to vertices in $G-T$.*

Now we study the graph G when 4-chords exist. We first remark the following:

Lemma 3.13. *Let G be a graph of girth n such that $v(G) \leqslant 3n-7$. Then G has neither cycle with* 2 *parallel* 2*-chord nor a cycle containing a* 3*-chord with a vertex joined to a vertex of the cycle distinct from the ends of the chord.*

Corollary 3.14. *Let G be a graph of girth n such that $v(G) \leqslant 3n-7$. Then G has no cycles with* 4 *free* 2*-chords.*

Lemma 3.15. *Let G be a graph of girth $n \geqslant 30$ such that $v(G) \leqslant 3n-7$. If G contains a cycle C with at least $3n-24$ vertices, then G contains a* 2*-vertex.*

Lemma 3.16. *Let G be a graph of girth $n \geqslant 30$ such that $v(G) \leqslant 3n-7$. Then one of the statements of Lemma* 3.12 *holds.*

Proof of Theorem 3.6. It is sufficient to show that G may have an orientation satisfying Beineke theorem. It is obvious if G contains exactly one cycle or if G is a good small graph. If G contains a 2-vertex v, we argue by induction by remarking that any orientation of $G-v$ respecting Beineke theorem can be extended to G. By the above lemma we have only to study the case where G contains a tree T with exactly two vertices x and y joined to two vertices x' and y' in $G-T$ respectively, such that x and y are non adjacent and belong to G_2. Let P be the path in T of ends x and y. Since x and y are non adjacent, then P contains a vertex w distinct from x and y. By induction $G-T$ may have an orientation L respecting Beineke theorem. If the edges xx' and yy' are oriented in different ways with respect to P, suppose that $(x',x) \in E(L)$ and $(y,y') \in E(L)$, then we complete by orienting directly P from x to y. The remaining edges are oriented away from it. In the other case, suppose that $(x',x) \in E(L)$ and $(y',y) \in E(L)$, we orient directly the paths xw and yw from x to w and from y to w. The other remaining edges of T are oriented towards P. The digraph obtained in both cases is a line digraph, so G is functional graph.

To show that the established bound is the best possible, consider the small graph of Proposition 3.4. Using the calculation of Lemma 3.10, we may easily verify that $v(G) \geqslant 3n-6$. The equality is established if the lengths of the three paths are equal to $\frac{n}{2}$ for n even. This achieves the proof of Theorem 3.6. □

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