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Mathematical Problems in Mechanics

Orbital stability and singularity formation for Vlasov–Poisson systems

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Abstract

We study the gravitational Vlasov–Poisson system in dimension $N=3$ and $N=4$ and consider the problem of nonlinear stability of steady states solutions within the framework of concentration compactness techniques. In dimension $N=3$ where the problem is subcritical, we prove the orbital stability in the energy space of the polytropes which are ground state type stationary solutions, which improves the already published results for this class. In dimension $N=4$ where the problem is L^1 critical, polytropic steady states are obtained following Weinstein [M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (1983) 567–576] by minimizing a suitable Gagliardo Nirenberg type inequality. Now a striking feature is the existence of a pseudo-conformal symmetry which allows us to derive explicit critical mass finite time blow up solutions. This is to our knowledge the first result of description of a singularity formation in a Vlasov setting. A general mass concentration phenomenon is eventually obtained for finite time blow up solutions. ***To cite this article: M. Lemou et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Stabilité orbitale et formation de singularité pour des systèmes de Vlasov–Poisson. Nous considérons le système de Vlasov–Poisson gravitationnel en dimensions $N=3$ et $N=4$ et replaçons l'étude de la stabilité non linéaire des états stationnaires dans le cadre des techniques de concentration compacité. En dimension $N=3$ où le problème est sous-critique, nous démontrons la stabilité orbitale dans l'espace d'énergie des polytropes qui sont des solutions stationnaires de type *ground state*, ce qui améliore pour cette classe les résultats déjà publiés. En dimension $N=4$ où le problème est L^1 critique, les polytropes sont obtenus dans la lignée de Weinstein [M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (1983) 567–576] comme minimiseurs d'une inégalité de type Gagliardo–Nirenberg. Un fait remarquable est maintenant l'existence d'une symétrie conforme qui nous permet d'écrire des solutions explosives explicites de masse critique. Ceci constitue à notre connaissance le premier résultat de description d'une formation de singularité dans le

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cadre des équations cinétiques de type Vlasov. Un résultat général de concentration de masse est enfin obtenu pour les solutions explosives. ***Pour citer cet article : M. Lemou et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Nous considérons le système de Vlasov–Poisson gravitationnel (1) en dimensions $N = 3$ et $N = 4$. En dimension $N = 3$, ce système décrit l'évolution cinétique d'un système stellaire soumis à l'action de son propre champ gravitationnel. Les systèmes auto-gravitants en dimension $N \geqslant 4$ sont également étudiés dans la littérature physique [3].

Notre but est de replacer l'étude de la stabilité non linéaire des solutions stationnaires dans le cadre des techniques de concentration compacité introduites par P.-L. Lions dans [10]. Nous obtenons une analogie formelle remarquable entre ce système et l'équation de Schrödinger non linéaire focalisante aussi bien quant à la stabilité non linéaire des solitons qu'à la structure formelle du problème d'explosion.

Dans une première partie, nous considérons le cas sous-critique $N = 3$. Plusieurs travaux ont été consacrés à l'étude de l'existence et à la stabilité non linéaire de solutions stationnaires de (1) ; nous renvoyons à [4] et ses références pour l'historique du problème. Une large classe d'états stationnaires a en particulier été obtenue via un problème de minimisation de type *energy-Casimir* dans [7], mais la stratégie utilisée ne permet d'obtenir la stabilité non linéaire des minimiseurs qu'au sens d'une distance locale. Notre approche est différente. Considérant $\mathcal{H}(f)$ le Hamiltonien de (1) donné par (2), nous démontrons dans le cadre des techniques de concentration compacité de [10] que pour une large classe de fonctions convexes j, les suites minimisantes du problème

$$\inf\left\{\mathcal{H}(f);\ |f|_{L^1} = M_1,\ \left|j(f)\right|_{L^1} = M_j\right\}$$

sont relativement compactes à translation près *dans l'espace d'énergie* (Théorème 2.1). L'unicité du minimiseur est un problème ouvert en toute généralité mais est assurée dans le cas homogène $j(f) = f^p$ pour lequel le minimiseur est un polytrope. La structure Hamiltonienne de (1) permet alors, dans la lignée de [2], de conclure à la stabilité orbitale des polytropes *dans l'espace d'énergie* (Corollaire 2.2), améliorant ainsi les résultats de [6,7,4] pour cette classe.

Dans une deuxième partie, nous nous focalisons sur le cas L^1 critique de la dimension $N = 4$. Les polytropes sont ici obtenus dans la lignée de [15] comme minimiseurs d'une inégalité de type Gagliardo–Nirenberg. Un corollaire classique est l'obtention d'un critère optimal de globalité des solutions faibles dans l'espace d'énergie (Corollaire 3.1). Notons que, comme cela est souvent le cas, l'existence de solutions explosives en temps fini est connue depuis longtemps via un argument obstructif de type viriel, voir [5] ; néanmoins, aucune information dynamique sur la formation de singularité n'est ainsi obtenue. Un fait remarquable est maintenant l'existence d'une symétrie conforme explicite qui, appliquée au polytrope, permet d'obtenir *des solutions explosives de masse critique* avec une formation de singularité totalement explicite. La structure du problème d'explosion est donc à cet égard similaire à celle pour l'équation de Schrödinger non linéaire L^2 critique (voir par exemple [11] et ses références). Nous démontrons enfin la stabilité orbitale du polytrope comme profil d'explosion (Théorème 3.2) et un phénomène général de concentration de la masse pour les solutions explosives de (1) (Théorème 3.3) dans la lignée de [13].

Cette Note est une version abrégée de [9].

1. Introduction: Hamiltonian structure and symmetries

We study the gravitational Vlasov–Poisson system in dimensions $N = 3$ and 4, the unknown being the distribution function $f(t, x, v) \geqslant 0$ with $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$:

$$\text{(VPG)}\quad \begin{cases} f_t + v\cdot\nabla_x f - E\cdot\nabla_v f = 0,\\ f(t=0,x,v) = f_0(x,v)\geqslant 0,\end{cases}\quad\text{with}\begin{cases}\Delta_x\phi = \rho, & \phi(t,x)\to 0 \quad\text{as } |x|\to+\infty,\\ E=\nabla_x\phi, & \rho(t,x)=\int_{\mathbb{R}^N} f(t,x,v)\,\mathrm{d}v.\end{cases}\tag{1}$$

For a given nonnegative distribution function $f(x,v)$, ρ_f denotes the corresponding density, ϕ_f is the Poisson potential and E_f is the corresponding force field, these quantities being defined by

$$\rho_f(x) = \int_{\mathbb{R}^N} f(x,v)\,\mathrm{d}v,\quad \phi_f(x) = -\frac{1}{N(N-2)\omega_N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}\rho_f(y)\,\mathrm{d}y,\quad E_f = \nabla_x\phi_f$$

where ω_N is the volume of the unit N-ball. Let $p_{\text{crit}} = (6N-N^2)/(4N+4-N^2)$ and define for $p\in[p_{\text{crit}},+\infty]$ the energy space

$$\mathcal{E}_p = \big\{f\geqslant 0 \text{ with } |f|_{\mathcal{E}_p} = |f|_{L^1} + |f|_{L^p} + \big||v|^2 f\big|_{L^1} < +\infty\big\}.$$

The existence of weak solutions locally in time to (1) in the energy space with possible extra assumptions on the initial data has been discussed in several works (see for instance [1,8], and references therein). These weak solutions satisfy an upper bound on the Hamiltonian and the L^q norm:

$$\mathcal{H}\big(f(t)\big) = \big||v|^2 f(t)\big|_{L^1} - \big|E_f(t)\big|^2_{L^2}\leqslant\mathcal{H}(f_0),\quad \big|f(t)\big|_{L^q}\leqslant|f_0|_{L^q},\quad \forall 1\leqslant q\leqslant p\tag{2}$$

which correspond to the Hamiltonian structure of (1). We have in $\mathcal{E}_p$ the standard interpolation estimate relating the kinetic and the potential energy:

$$\forall p\in[p_{\text{crit}},+\infty],\quad |E_f|^2_{L^2}\leqslant C_p\big||v|^2 f\big|^{\frac{N-2}{2}}_{L^1}|f|^{\frac{4N+4-N^2}{2N(p-1)}(p-p_{\text{crit}})}_{L^1}|f|^{\frac{p(N-2)}{N(p-1)}}_{L^p}.\tag{3}$$

System (1) has a large group of symmetries in the energy space $\mathcal{E}_p$: if $f(t,x,v)$ solves (1), then $\forall(t_0,x_0,v_0)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, $\forall(\lambda_0,\mu_0)\in\mathbb{R}^+_*$, so does

$$\frac{\mu_0^{N-2}}{\lambda_0^2} f\left(\frac{t+t_0}{\lambda_0\mu_0},\frac{x+x_0+v_0t}{\lambda_0},\mu_0(v+v_0)\right).\tag{4}$$

In dimension $N=3$, weak solutions are known to be global and bounded in $\mathcal{E}_p$ since the bound on the Hamiltonian and the L^p norm together with (3) and $\frac{N-2}{2}<1$ imply a uniform upper bound on the kinetic energy. On the contrary in dimension $N=4$, it is well known from the virial identity that blow up can occur [5], but this obstructive argument provides no insight into the singularity formation. Note that (1) is L^1 critical in the sense that *the L^1 norm is conserved and all symmetries are L^1 isometries* – at least for $\mu_0=\lambda_0$ in scaling formula (4).

In dimension $N=3$, a number of works have been devoted to the existence and the nonlinear stability of steady states solutions to (1), we refer to [6,4,7] and references therein for a history of the problem. In particular, a large class of steady states has been obtained in [7] as minimizers of an energy-Casimir problem: let $M>0$ and j be some convex function, then

$$\inf\big\{\mathcal{H}(f);\ f\geqslant 0,\ |f|_{L^1}+\big|j(f)\big|_{L^1}=M\big\}\tag{5}$$

is attained on a steady state solution to (1). Moreover, using a weak form of concentration compactness techniques, a dynamical nonlinear stability statement is proven. Let us insist onto the following points: (i) First, strategy like (5) would fail to build steady states for $N=4$ due to the L^1 scaling invariance. In the special case $j(f)=f^k$, there are in fact simpler ways of deriving a minimization problem solved by the steady state than (5) which suffers from the lack of compactness due to translation invariance. In particular, an adaptation of the approach in [15] will be successful for $N=3$ and 4. (ii) The nonlinear stability statement of the obtained steady states in [6,7,4] is measured in terms of a distance which under some very specific constraints on the perturbation is proved to control the L^2 norm. One of the difficulties the authors are confronted with is the *weak L^p* convergence only of the minimizing sequences of (5).

Our aim in this Note is to view the problem through the standard concentration compactness techniques introduced in [10] and to derive the natural orbital stability statements in this framework, both in dimensions $N=3$ and 4.

2. Orbital stability of the polytropes

In dimension $N=3$, we claim following [10] that the minimizing sequences of a large class of minimization problems are up to a translation shift relatively compact in the energy space and then the infimum is attained on a steady state solution to (1).

Theorem 2.1 (Compactness of the minimizing sequences). *Let* $N=3$. *Let* j *be a strictly convex continuous non-negative function on* $\mathbb{R}_+$ *such that one of the following holds*:

(i) $j(t)=t^p$ *for some* $p_{\text{crit}}<p<+\infty$;
(ii) *there exists* $\frac{3}{2}<p_1\leqslant p_2<+\infty$ *such that* $\forall t\geqslant 0$, $\forall b\geqslant 1$, $b^{p_1}j(t)\leqslant j(bt)\leqslant b^{p_2}j(t)$.

Given $M_1, M_j>0$, *let* $\mathcal{F}(M_1,M_j)=\{f\in\mathcal{E}_p$ *with* $\int_{\mathbb{R}^6} f=M_1$, $\int_{\mathbb{R}^6} j(f)=M_j\}$, *and*

$$I(M_1,M_j)=\inf\bigl\{\mathcal{H}(f);\ f\in\mathcal{F}(M_1,M_j)\bigr\}. \tag{6}$$

Then for every minimizing sequence $(f_n)_{n\geqslant 1}$, *there exists* $(y_n)_{n\geqslant 1}$ *in* $\mathbb{R}^3$ *such that* $f_n(x+y_n,v)$ *is relatively compact in the energy space* $\mathcal{E}_p$ *and converges to a minimizer.*

A similar statement is also obtained in the limiting case $p=+\infty$, *i.e.* when the constraint $\int_{\mathbb{R}^6} j(f)=M_j$ is replaced by $|f|_{L^\infty}=M_\infty$ as in [4].

The question of uniqueness of the minimizer for (6) is open in general, but is straightforward in the homogeneous case $j(f)=f^p$ in which case the minimizer is the so called polytrope

$$Q_p(x,v)=(-1-e)_+^{1/(p-1)}=\max(0,-1-e)^{1/(p-1)}, \tag{7}$$

where $e=\frac{|v|^2}{2}+\phi_p$ and ϕ_p is the unique radial solution to

$$\frac{1}{r^{N-1}}\frac{\mathrm{d}}{\mathrm{d}r}\bigl(r^{N-1}\phi_p'\bigr)+\gamma_{N,p}(-1-\phi_p)_+^{1/(p-1)+N/2}=0,\quad \phi_p(r)\to 0\quad \text{as } r\to+\infty, \tag{8}$$

where $p_{\text{crit}}<p\leqslant+\infty$ and $\gamma_{N,p}=N\omega_N\int_0^1(2t)^{(N-2)/2}(1-t)^{1/(p-1)}\,\mathrm{d}t$.

Following [2], the Hamiltonian structure of (1) and Theorem 2.1 classically imply the orbital stability *in the energy space* of the polytropes.

Corollary 2.2 (Orbital stability of Q_p in $\mathcal{E}_p$). *Let* $N=3$ *and* $p_{\text{crit}}<p\leqslant+\infty$. *Then for all* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that the following holds true. Let* $f_0\in\mathcal{E}_p$ *with*

$$\mathcal{H}(f_0)-\mathcal{H}(Q_p)\leqslant\delta(\varepsilon),\quad |f_0|_{L^1}\leqslant|Q_p|_{L^1}+\delta(\varepsilon),\quad \textit{and}\quad |f_0|_{L^p}\leqslant|Q_p|_{L^p}+\delta(\varepsilon),$$

and let $f(t)\in L^\infty(\mathbb{R}_+,\mathcal{E}_p)$ *a weak solution to* (1) *with initial data* f_0, *then there exists a translation shift* $x(t)\in\mathbb{R}^N$ *such that* $\forall t\in[0,+\infty)$,

$$\bigl|f\bigl(t,x+x(t),v\bigr)-Q_p\bigr|_{\mathcal{E}_p}<\varepsilon\quad \textit{if } p<+\infty,\quad \textit{or}$$
$$\sup_{\|\psi\|_{L^\infty}=1}\bigl|\bigl(f(t,x+x(t),v)-Q_p,\psi\bigr)\bigr|<\varepsilon\quad \textit{if } p=+\infty.$$

Very recently and independently from this work, Sánchez and Soler, [14], have obtained the orbital stability of Q_p in dimension $N=3$ in $L^1\cap L^p$, $\frac{9}{7}<p<+\infty$.

3. Singularity formation and mass concentration for $N = 4$

In dimension $N = 4$, strategy (6) to build polytropes fails due to the L^1 scaling invariance. Following [15], the four-dimensional polytrope Q_p given by (7) and (8) with $N = 4$ and $2 < p \leqslant +\infty$ may be obtained as a best constant in Gagliardo Nirenberg type inequality (3), or equivalently:

$$\forall f \in \mathcal{E}_p \quad \mathcal{H}(f) \geqslant \left||v|^2 f\right|_{L^1}\left(1 - \left(\frac{|f|_{L^1}|f|_{L^p}^{p/(p-2)}}{|Q_p|_{L^1}|Q_p|_{L^p}^{p/(p-2)}}\right)^{(p-2)/(2(p-1))}\right).$$

This implies a global wellposedness criterion for the Cauchy problem of (1).

Corollary 3.1 (Global wellposedness criterion). *Let $N = 4$ and $p_{\text{crit}} < p \leqslant +\infty$. Let $f_0 \in \mathcal{E}_p \cap L^\infty$ with $|f_0|_{L^1}|f_0|_{L^p}^{p/(p-2)} < |Q_p|_{L^1}|Q_p|_{L^p}^{p/(p-2)}$, then there exists a global weak solution to* (1) *with initial data f_0.*

Now a remarkable fact is that this criterion is sharp, as it is the case for the L^2 critical nonlinear Schrödinger equation (see [15]). Indeed, (1) admits for $N = 4$ an additional pseudo-conformal symmetry: if $f(t,x,v)$ is a solution to (1), then $\forall a \in \mathbb{R}$, so is $f_a(t,x,v) = f(\frac{t}{1-at}, \frac{x}{1-at}, (1-at)v + ax)$. Applying this to the polytrope Q_p yields an explicit critical mass blow up solution to (1):

$$S_p(t,x,v) = Q_p\left(\frac{x}{1-t}, (1-t)v + x\right).$$

It blows up at time $T = 1$ with speed

$$\left||v|^2 S_p(t)\right|_{L^1}^{1/2} \sim \frac{C_p}{1-t} \quad \text{as } t \to 1$$

and leaves $\mathcal{E}_p$ by leaving L^1 and forming a Dirac mass:

$$\int_{\mathbb{R}^4} S_p(x,v)\,\mathrm{d}v \rightharpoonup |Q_p|_{L^1}\delta_{x=0} \quad \text{as } t \to 1$$

in the weak sense of measures.

We now claim the orbital stability of Q_p as a blow up profile.

Theorem 3.2 (Orbital stability of Q_p as a blow up profile). *Let $p_{\text{crit}} < p < \infty$. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that the following holds true. Let $f_0 \in \mathcal{E}_p$ with*

$$|f_0|_{L^1} \leqslant |Q_p|_{L^1} + \delta(\varepsilon), \qquad |f_0|_{L^p} \leqslant |Q_p|_{L^p} + \delta(\varepsilon)$$

and $f(t) \in \mathcal{E}_p$, $t \in [0,T)$, a weak solution to (1) *with initial data f_0 which blows up in finite time $0 < T < +\infty$ in the sense that $\lim_{t\to T} ||v|^2 f(t)|_{L^1} = +\infty$. Let $\lambda(t) = (||v|^2 Q_p|_{L^1}/||v|^2 f(t)|_{L^1})^{1/2}$, then there exists a translation shift $x(t) \in \mathbb{R}^N$ such that $\forall t \in [0,T)$,*

$$\left|f\left(t, \lambda(t)\big(x + x(t)\big), \frac{v}{\lambda(t)}\right) - Q_p\right|_{\mathcal{E}_p} < \varepsilon.$$

Let us insist onto the fact that this kind of nonlinear stability statement is obtained from purely variational arguments but is in the setting of the critical nonlinear Schrödinger equation the *starting point* of the recent fine dynamical analysis of the singularity formation in [11]. Eventually, we claim following [13] a mass concentration phenomenon for finite time blow up solutions to (1).

Theorem 3.3 (Mass concentration in L^1 and L^p). *Let $N=4$ and $p_{\text{crit}}<p<+\infty$. Let $f_0\in\mathcal{E}_p$ and $f(t)\in\mathcal{E}_p$, $t\in[0,T)$, a weak solution to* (1) *with initial data f_0. Assume that $f(t)$ blows up in finite time $0<T<+\infty$ in the sense that $\||v|^2 f(t)|_{L^1}\to+\infty$ as $t\to T$. Let $\rho_q(t,x)=(\int_{\mathbb{R}^4} f^q(t,x,v)\,\mathrm{d}v)^{1/q}$ for $q=1$ or $q=p$. Then there exists a translation shift $x(t)\in\mathbb{R}^N$ such that, for all $R>0$,*

$$M_1(R)=\liminf_{t\to T}\int_{|x-x(t)|<R}\rho_1(t,x)\,\mathrm{d}x,\qquad M_p(R)=\liminf_{t\to T}\biggl(\int_{|x-x(t)|<R}\rho_p^p(t,x)\,\mathrm{d}x\biggr)^{1/p}$$

satisfy

$$M_1(R)\bigl(M_p(R)\bigr)^{p/(p-2)}\geqslant|Q_p|_{L^1}|Q_p|_{L^p}^{p/(p-2)}.$$

Again, this result of variational nature is a preliminary step towards the description of the singularity formation. Recently, it was proven in [12] that small super critical mass finite time blow up solutions to the critical nonlinear Schrödinger equation focus exactly the quantized and universal amount of critical mass at blow up time. This type of question now arises naturally from Theorem 3.3 and is widely open for (1).

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