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Probability Theory

Riemannian connections and curvatures on the universal Teichmuller space

Hélène Airault [a,b]

[a] *INSSET, université de Picardie, 48, rue Raspail, 02100 Saint-Quentin, France*
[b] *Laboratoire CNRS UMR 6140 LAMFA, 33, rue Saint-Leu, 80039 Amiens, France*

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Abstract

We define Riemannian connections on the *universal Teichmuller space* $\mathcal{U}^\infty$. For the Levi-Civita's connection on $\mathcal{U}^\infty$, the Riemannian curvature tensor is well defined and the Ricci curvature is finite. We obtain several series of infinite dimensional operators which converge. ***To cite this article: H. Airault, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Connexions riemanniennes et courbures sur l'espace de Teichmuller universel. On définit plusieurs connexions riemanniennes sur *l'espace de Teichmuller universel* $\mathcal{U}^\infty$. Pour la connexion de Levi-Civita sur $\mathcal{U}^\infty$, le tenseur de courbure existe et la courbure de Ricci est finie. On obtient plusieurs séries d'opérateurs de l'espace de dimension infinie qui convergent. ***Pour citer cet article : H. Airault, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit $\mathrm{Diff}(S^1)$ le groupe des difféomorphismes du cercle qui préservent l'orientation et soit $\mathcal{H}$ le sous-groupe des transformations homographiques. Un difféomorphisme $e^{i\gamma(\theta)}$ s'identifie avec l'application $\gamma:\theta\to\gamma(\theta)$ *modulo* 2π. Soit $\mathrm{diff}(S^1)$ l'algèbre de Lie de $\mathrm{Diff}(S^1)$. L'algèbre de Lie de $\mathcal{H}$ est notée $\mathrm{su}(1,1)$, elle est engendrée par $\cos\theta$, $\sin\theta$, 1. Pour k un entier, $k\geqslant 0$, on pose $\alpha(k)=ak^3+bk$ où $a\geqslant 0$ et b est un nombre réel. Sur l'espace vectoriel V des séries de Fourier $u(\theta)=\sum_{k\geqslant 0}a_k^u\cos(k\theta)+b_k^u\sin(k\theta)$ telles que $\|u\|^2=\sum_{k\geqslant 1}(a_k^u)^2\alpha(k)+(b_k^u)^2\alpha(k)<+\infty$, on considère le produit scalaire

E-mail address: helene.airault@insset.u-picardie.fr (H. Airault).

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$$(u|v)=\sum_{k\geqslant 1} a_k^u a_k^v \alpha(k)+b_k^u b_k^v \alpha(k). \tag{1}$$

Comme $(w'|u)=\sum_{k\geqslant 2}(a_k^u b_k^w - a_k^w b_k^u)\,k\,\alpha(k)$, on a $(w'|u)=-(w|u')$. Pour $u,\,v\in V$,

$$[u,v](\theta)=u(\theta)v'(\theta)-u'(\theta)v(\theta). \tag{2}$$

Lorsque $\alpha(k)=k^3-k$ et pour $u,\ v,\ w$ dans le sous-espace vectoriel V_0 engendré par $\{\cos(k\theta);\sin(k\theta)\}_{k\geqslant 2}$, on démontre l'identité remarquable

$$\big(Ju\mid[v,w]\big)+\big(Jv\mid[w,u]\big)+\big(Jw\mid[u,v]\big)=0, \tag{3}$$

où J est la transformation de Hilbert, $J\cos k\theta=\sin k\theta$ et $J\sin k\theta=-\cos k\theta,\ k\geqslant 1$. Soit π la projection de $\mathrm{diff}(S^1)$ sur V_0. Lorsque $u,v,w\in\mathrm{diff}(S^1)$, on définit la connexion de Levi-Civita sur le groupe quotient $\mathcal{H}\backslash\mathrm{Diff}(S^1)$ par (cf. [12, vol. 2, p. 201])

$$\Gamma_{\mathrm{LC}}(u,v,w)=\big(\Gamma_{\mathrm{LC}}(v)u\mid w\big)=\frac{1}{2}\big[\big([w,u]\mid\pi v\big)+\big([w,v]\mid\pi u\big)-\big([u,v]\mid\pi w\big)\big]. \tag{4}$$

La courbure de cette connexion est égale à celle de la connexion obtenue dans [6]. On étudie la convergence de séries formées par des opérateurs diagonaux dans la base orthonormale $\{c_k=\frac{\cos(k\theta)}{\sqrt{k^3-k}};\ s_k=\frac{\sin(k\theta)}{\sqrt{k^3-k}}\}_{k\geqslant 2}$ et issues de différentes connexions. Soit $\Gamma_1(v)u=[u,v]$. La série d'opérateurs $\sum_{j\geqslant 2}\Gamma_{\mathrm{LC}}(c_j)^2+\Gamma_{\mathrm{LC}}(s_j)^2$ converge et la série

$$\Lambda=\sum_{j\geqslant 2}(\Gamma_{\mathrm{LC}}+\Gamma_1)(c_j)\Gamma_{\mathrm{LC}}(c_j)+(\Gamma_{\mathrm{LC}}+\Gamma_1)(s_j)\Gamma_{\mathrm{LC}}(s_j) \tag{5}$$

définit un opérateur de courbure borné.

1. Introduction

Among the difficulties of geometry in infinite dimension are the importance of a good choice of coordinate system and the divergence of series associated to the operation of contraction. Infinite-dimensional geometry appears naturally in Probability Theory [1,2,7] and in Mathematical Physics [6,5,11]. The *universal Teichmuller space* $\mathcal{U}^\infty$ is the space of C^∞ Jordan curves of the complex plane. In our program, $\mathcal{U}^\infty$ will appear as the *skeleton* of canonical probability measures which will be carried by a larger space than $\mathcal{U}^\infty$ (see [4,10]). To $\gamma\in\mathcal{U}^\infty$, correspond two univalent functions $f_\gamma^+,\ f_\gamma^-$ sending the inside and the outside of the unit disk on the two regions delimited by γ; then $f_\gamma^+,\ f_\gamma^-$ are C^∞ functions on the circle S^1. Therefore, $g_\gamma(\theta):=[(f_\gamma^+)^{-1}\circ f_\gamma^-]\exp(\mathrm{i}\theta)$ is a diffeomorphism of the circle. Denote $\mathrm{Diff}(S^1)$ the group of C^∞, orientation preserving diffeomorphisms of the circle. Let $\mathcal{H}$ be the subgroup of Möbius transformations of the disk considered as operating on S^1; then $f_\gamma^+,\ f_\gamma^-$ are defined up to a Möbius transformation. The previous construction defines an injective map $\mathcal{U}^\infty\mapsto\mathcal{H}\backslash\mathrm{Diff}(S^1)/\mathrm{Rot}(S^1)$. The Beurling–Ahlfors theory of conformal welding shows that this map is surjective. The Lie algebra $\mathrm{diff}(S^1)$ is the set of C^∞ vector fields on the circle, they are identified with the C^∞ functions on the circle. The Lie algebra $\mathrm{su}(1,1)$ of $\mathcal{H}$ is the linear subspace of $\mathrm{diff}(S^1)$ having for basis $1,\ \cos\theta,\ \sin\theta$. It has been shown in [3] that there exists a *unique* scalar product on $\mathrm{diff}(S^1)$ which is invariant under the adjoint action of $\mathrm{su}(1,1)$. The associated metric defines a *canonical Riemannian structure on* $\mathcal{U}^\infty$. An orthonormal basis for this metric at identity is $\{\frac{1}{\sqrt{k^3-k}}\cos k\theta,\ \frac{1}{\sqrt{k^3-k}}\sin k\theta\},\ k>1$. In this Note,

(i) we prove the identity (3) which is specific to the metric $\alpha(k)=k^3-k$,
(ii) we introduce different Riemannian connections on $\mathcal{U}^\infty$, among them the Levi-Civita connection (4) which commutes with the Hilbert transform J.

This leads to the Riemannian curvature and Ricci curvature of $\mathcal{U}^\infty$.

2. The geometry of $\mathcal{H}\backslash \mathrm{Diff}(S^1)$

We calculate on $\mathrm{Diff}(S^1)$ *modulo* composition on the left by homographic transformations. Let $\gamma \in \mathrm{Diff}(S^1)$ and $u(\theta) \in V$. To obtain right invariant vector fields, we define for small $\epsilon > 0$, $\gamma_\epsilon(\theta) = \gamma(\theta) + \epsilon u(\gamma(\theta)) = (\exp(\epsilon u) \circ \gamma)(\theta) + \mathrm{o}(\epsilon^2)$, then $X_u(\gamma) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\gamma_\epsilon = u \circ \gamma$ is right invariant since for the right translation $\mathcal{R}(\gamma) = \gamma \circ \gamma_1$, we have $\mathrm{d}\mathcal{R}\,(\gamma)[X_u(\gamma)] = X_u(\gamma \circ \gamma_1) = X_u(\gamma) \circ \gamma_1$. In the same way, we put $\gamma_\epsilon(\theta) = \gamma(\theta + \epsilon u(\theta)) = (\gamma \circ \exp(\epsilon u))(\theta) + \mathrm{o}(\epsilon^2)$, then $Y_u(\gamma)(\theta) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\gamma_\epsilon(\theta) = \gamma'(\theta)u(\theta)$ where $\gamma'(\theta)$ is the derivative of γ with respect to θ, is left invariant. For a vector field $Z(\gamma)$, the parallel transport to the right is given by $Z(\gamma)(\theta) \to (Z(\gamma) \circ \gamma_1)(\theta)$ and the parallel transport to the left is given by $Z(\gamma)(\theta) \to \gamma_1'(\theta)Z(\gamma)(\theta)$. Let $\mathcal{L}\,(\gamma) = \gamma_1 \circ \gamma$ be the left translation. We define $\mathrm{Ad}(\gamma): V \to V$ by $\mathrm{Ad}(\gamma)(u)(\theta) = \gamma'(\gamma^{-1}(\theta))u(\gamma^{-1}(\theta))$. Then $\mathrm{Ad}(\gamma^{-1})(u)(\theta) = \frac{u(\gamma(\theta))}{\gamma'(\theta)}$. For a rotation of angle ϕ, $\theta + \phi = r_\phi(\theta)$, $\mathrm{Ad}(r_\phi)(u) = u(\theta + \phi)$. Let $u, v \in V$ and consider the vector fields $X_u(\gamma) = u \circ \gamma$ and $X_w(\gamma) = w \circ \gamma$; the bracket is given by $[X_u(\gamma), X_v(\gamma)] = [u, v] \circ \gamma$. For $Y_u(\gamma) = \gamma'(\cdot)u(\cdot)$, then $[Y_u(\gamma), Y_v(\gamma)] = \gamma'(\cdot)[u, v]$.

3. The Hilbert transform

Let $J\cos k\theta = \sin k\theta$ and $J\sin k\theta = -\cos k\theta$, $k \geqslant 1$. The Hilbert transform J allows to sort out the $\cos k\theta$, $\sin p\theta, \ldots$ according to whether $k > p$ or $k < p$. With (2), we obtain

$$\begin{aligned} &[J\cos k\theta, J\cos p\theta] - [\cos k\theta, \cos p\theta] = (p-k)\sin(p+k)\theta, \\ &[J\sin k\theta, J\cos p\theta] - [\sin k\theta, \cos p\theta] = (k-p)\cos(p+k)\theta \end{aligned} \tag{6}$$

and for $X = \sin k\theta$, or $\cos k\theta$ and $Y = \sin p\theta$, or $\cos p\theta$, with $p, k \geqslant 2$,

$$[JX, JY] - [X, Y] = J\big([X, JY] + [JX, Y]\big). \tag{7}$$

Since $J^2 = -\,\mathrm{Id}$, then (7) can be written $[X, JY] + [JX, Y] = J[X, Y] - J[JX, JY]$. Define

$$\begin{aligned} &A(X,Y) = J[X,Y] - [JX,Y] \quad \text{and} \quad B(X,Y) = J[X,Y] - [X,JY], \\ &\{X,Y\} = \frac{1}{2}\big([X,JY] + [JX,Y]\big) = \frac{1}{2}J\big([X,Y] - [JX,JY]\big). \end{aligned} \tag{8}$$

When $X \neq Y$, we have $B(X, Y) = JB(X, JY)$ as well as $A(X, Y) = JA(JX, Y)$ and $A(X, JY) + A(JX, Y) = 0$. Eq. (6) is equivalent to any of the two following $A(JX, JY) = A(X, Y)$ or $B(JX, JY) = B(X, Y)$. For $\{\,,\,\}$, we have the Jacobi identity

$$\big\{X, \{Y, Z\}\big\} + \big\{Y, \{Z, X\}\big\} + \big\{Z, \{X, Y\}\big\} = 0 \tag{9}$$

and $\{JX, Y\} = J\{X, Y\} = \{X, JY\}$. With (7), we deduce

$$\begin{aligned} &J\big([JX,JY] + [X,Y]\big) - \big([JX,Y] - [X,JY]\big) = 2A(X,Y), \\ &J\big([JX,JY] + [X,Y]\big) + \big([JX,Y] - [X,JY]\big) = 2B(X,Y). \end{aligned} \tag{10}$$

4. The scalar product on $\mathrm{diff}(S^1)/\mathrm{su}(1,1)$

With the scalar product (1), it was remarked in [3] that $\alpha(k) = \mathit{constant} \times (k^3 - k)$ when $\mathrm{Ad}(h)$ is a unitary operator for any homographic transformation h. In fact, let $u_{2k}(\theta) = \cos(k\theta)$, $u_{2k+1}(\theta) = \sin(k\theta)$, the condition $([u, u_p] \mid u_q) + (u_p \mid [u, u_q]) = 0$, $\forall p, q \geqslant 2$ and $u = 1, \cos(\theta), \sin(\theta)$ is equivalent to $(1-k)\alpha(1+k) + (2+k)\alpha(k) = 0$ if $k \neq 2$ and $-\alpha(3) + 4\alpha(2) = 0$ which determines completely $\alpha(k) = \mathit{constant} \times (k^3 - k)$. In that case, for $m, n, p, j \in Z$, $(m-n)\alpha(p) + (n-p)\alpha(m) + (p-m)\alpha(n) = 0$ if $m + n + p = 0$ and

$$(j-k)(m+p)\alpha(j+k)-(p-k)(m+j)\alpha(p+k)$$
$$=(j-p)\big[(k+j+2p)\alpha(k+j)+(p-k)\alpha(j+p)\big] \quad \text{if } m=k+j+p.$$

Lemma 4.1. *If $\alpha(k) = constant \times (k^3 - k)$, for u, v, w in $\mathrm{diff}(S^1)$, we have*

$$\begin{aligned}&\big(\sin k\theta \mid [\cos j\theta, \cos p\theta]\big) + \big(\sin j\theta \mid [\cos p\theta, \cos k\theta]\big) + \big(\sin p\theta \mid [\cos k\theta, \cos j\theta]\big) = 0,\\ &\big(\cos k\theta \mid [\sin j\theta, \cos p\theta]\big) + \big(\cos j\theta \mid [\cos p\theta, \sin k\theta]\big) + \big(\sin p\theta \mid [\sin j\theta, \sin k\theta]\big) = 0.\end{aligned} \tag{11}$$

We verify (11) as follows. let δ_j^p be the Kronecker symbol, we add

$$\begin{aligned}
2\big(\sin k\theta \mid [\cos j\theta, \cos p\theta]\big) &= (j+p)\alpha(k)\delta_j^{k+p} + (j-p)\alpha(k)\delta_k^{j+p} - (j+p)\alpha(k)\delta_p^{k+j},\\
2\big(\sin j\theta \mid [\cos p\theta, \cos k\theta]\big) &= (p-k)\alpha(j)\delta_j^{k+p} - (k+p)\alpha(j)\delta_k^{j+p} + (k+p)\alpha(j)\delta_p^{k+j},\\
2\big(\sin p\theta \mid [\cos k\theta, \cos j\theta]\big) &= -(k+j)\alpha(p)\delta_j^{k+p} + (k+j)\alpha(p)\delta_k^{j+p} + (k-j)\alpha(p)\delta_p^{k+j},\\
2\big(\cos k\theta \mid [\sin j\theta, \cos p\theta]\big) &= -(j+p)\alpha(k)\delta_j^{k+p} - (j-p)\alpha(k)\delta_k^{j+p} - (j+p)\alpha(k)\delta_p^{k+j},\\
2\big(\cos j\theta \mid [\cos p\theta, \sin k\theta]\big) &= (k-p)\alpha(j)\delta_j^{k+p} + (k+p)\alpha(j)\delta_k^{j+p} + (k+p)\alpha(j)\delta_p^{k+j},\\
2\big(\sin p\theta \mid [\sin j\theta, \sin k\theta]\big) &= (k+j)\alpha(p)\delta_j^{k+p} - (k+j)\alpha(p)\delta_k^{j+p} + (k-j)\alpha(p)\delta_p^{k+j}.
\end{aligned}$$

From (11), we deduce that $(u \mid [Jv, Jw]) + (v \mid [Jw, Ju]) + (w \mid [Ju, Jv]) = 0$ or equivalently (3) is true for u, v, w of the form $\cos k\theta$ or $\sin j\theta$. We use the linearity in each variable to prove (3) in its generality.

From (3), we obtain more identities, let $\{u, v\} = \frac{1}{2}([u, Jv] + [Ju, v])$ as in (8), then

$$\begin{aligned}&\big(u \mid \{v, w\}\big) + \big(v \mid \{w, u\}\big) + \big(w \mid \{u, v\}\big) = \frac{1}{2}\Big(\big(Ju \mid [Jv, Jw]\big) + \big(Jv \mid [Jw, Ju]\big) + \big(Jw \mid [Ju, Jv]\big)\Big),\\ &\big(Ju \mid \{v, w\}\big) + \big(Jv \mid \{w, u\}\big) + \big(Jw \mid \{u, v\}\big) = \frac{1}{2}\Big(\big(u \mid [v, w]\big) + \big(v \mid [w, u]\big) + \big(w \mid [u, v]\big)\Big).\end{aligned} \tag{12}$$

5. The Levi-Civita's transfer field and related fields

We follow [12, vol. 1, p. 160 and vol. 2, p. 201], [7] and [2, p. 452]. For $u, v, w \in \mathrm{diff}(S^1)$, we define $\Gamma_{\mathrm{LC}}(u, v, w)$ by (4). We denote $\Gamma_{\mathrm{LC}}(u, v, w) = (\Gamma_{\mathrm{LC}}(v)u \mid \pi w) = (\Gamma_{\mathrm{LC}})_{vu}^w$. Since $\Gamma_{\mathrm{LC}}(u, v, w)$ is antisymmetric in (u, w), the adjoint Γ_{LC}^* satisfies $\Gamma_{\mathrm{LC}}^* = -\Gamma_{\mathrm{LC}}$. With (3), we deduce that *the connection Γ_{LC} preserves the complex structure, $\Gamma_{\mathrm{LC}}(u, v, w) = \Gamma_{\mathrm{LC}}(Ju, v, Jw)$*. The transfer field Γ_{LC} is the half sum of two antisymmetric transfer fields

$$\Gamma_{\mathrm{LC}} = \frac{1}{2}[\Gamma_3 + \Gamma_5], \quad \text{with } \Gamma_3(u, v, w) = \big[\big([w, u] \mid \pi v\big),\ \Gamma_5(u, v, w) = \big([w, v] \mid \pi u\big) - \big([u, v] \mid \pi w\big)\big]. \tag{13}$$

For $\Gamma = \Gamma_3, \Gamma_5$, $\Gamma(v)J \neq J\Gamma(v)$. For $u, v, w \in \mathrm{diff}(S^1)$, we denote $\Gamma_{vu}^w = \Gamma(u, v, w)$ and we define

$$\begin{aligned}\Gamma_1(u, v, w) &= \big([u, v] \mid \pi w\big) = \big(\Gamma_1(v)u \mid \pi w\big) = (\Gamma_1)_{vu}^w,\\ \Gamma_2(u, v, w) &= \big([v, w] \mid \pi u\big) = \big(\Gamma_2(v)u \mid \pi w\big) = (\Gamma_2)_{vu}^w,\\ \Gamma_4(u, v, w) &= \big([w, u] \mid \pi v\big) + \big([w, v] \mid \pi u\big) = \big(\Gamma_4(v)u \mid \pi w\big) = (\Gamma_4)_{vu}^w,\end{aligned} \tag{14}$$

then $\Gamma_{\mathrm{LC}} = \frac{1}{2}[\Gamma_4 - \Gamma_1]$ and $\Gamma_5 = -(\Gamma_1 + \Gamma_2)$. Another connection also preserves the complex structure, see [6],

$$\Gamma_A(u, v, w) = \Gamma_{\mathrm{LC}}(Ju, Jv, Jw) = \big(\Gamma_A(v)u \mid w\big) = (\Gamma_A)_{vu}^w. \tag{15}$$

Let $A(v,w)$ and $B(v,w)$ as in (8). Then

$$\begin{aligned}&\big(B(v,w)\mid u\big)-\big(B(v,u)\mid w\big)=\big(Jv\mid[w,u]\big)+\big([Jw,v]\mid u\big)-\big([Ju,v]\mid w\big),\\&\big(A(v,w)\mid u\big)-\big(A(v,u)\mid w\big)=\big(\Gamma_A(Jv)u\mid w\big),\\&J\Gamma_A(Jv)u+\Gamma_A(v)u=2JB(u,v).\end{aligned}\tag{16}$$

To prove the last identity, we remark with (3) that

$$\big(\Gamma_{\rm LC}(Jv)u\mid Jw\big)-\big(\Gamma_{\rm LC}(v)u\mid w\big)=2\big(J[u,Jv]+[u,v]\mid w\big)=-2\big(JB(u,v)\mid w\big).\tag{17}$$

With (3), we obtain that $\Gamma_{\{\}}(u,v,w)=\frac{1}{2}[(\{w,u\}\mid\pi v)+(\{w,v\}\mid\pi u)-(\{u,v\}\mid\pi w)]$ satisfies

$$\Gamma_{\{\}}(u,v,w)=\frac{1}{2}\Gamma_{\rm LC}(Ju,Jv,Jw)+\frac{1}{2}\big([u,w]\mid Jv\big).\tag{18}$$

For $\Gamma_{\rm LC}$, the torsion is zero. Consider $\Gamma_A(u,v,w)=\Gamma_{\rm LC}(Ju,Jv,Jw)$, the torsion $T_{\Gamma_A}(u,v)$ is not zero, $(T_{\Gamma_A}(u,v)\mid w)=([Jv,Ju]\mid Jw)-(J[v,u]\mid Jw)$. The two tensors $\Gamma_{\rm LC}$ and Γ_A have the same curvature. *In complex coordinates, $\Gamma_{\rm LC}$, for $p,k\geqslant 2$, is given by*

$$\begin{aligned}&\Gamma_{\rm LC}(\mathrm{e}^{\mathrm{i}p\theta})\,\mathrm{e}^{\mathrm{i}k\theta}=\mathrm{i}\frac{(2p+k)\alpha(k)}{\alpha(p+k)}\,\mathrm{e}^{\mathrm{i}(p+k)\theta},\\&\Gamma_{\rm LC}(\mathrm{e}^{\mathrm{i}p\theta})\,\mathrm{e}^{-\mathrm{i}k\theta}=-\mathrm{i}(p+k)1_{k\geqslant p+2}\times\mathrm{e}^{-\mathrm{i}(k-p)\theta}=1_{k\geqslant p+2}\times[\mathrm{e}^{\mathrm{i}p\theta},\mathrm{e}^{-\mathrm{i}k\theta}],\\&\Gamma_{\rm LC}(\mathrm{e}^{-\mathrm{i}p\theta})\,\mathrm{e}^{\mathrm{i}k\theta}=\mathrm{i}(p+k)1_{k\geqslant p+2}\times\mathrm{e}^{\mathrm{i}(k-p)\theta}=1_{k\geqslant p+2}\times[\mathrm{e}^{-\mathrm{i}p\theta},\mathrm{e}^{\mathrm{i}k\theta}],\\&\Gamma_{\rm LC}(\mathrm{e}^{-\mathrm{i}p\theta})\,\mathrm{e}^{-\mathrm{i}k\theta}=-\mathrm{i}\frac{(2p+k)\alpha(k)}{\alpha(p+k)}\mathrm{e}^{-\mathrm{i}(p+k)\theta}.\end{aligned}\tag{19}$$

6. Composition of transfer fields: expression with symbols

For $j\geqslant 2$, let $\epsilon_{2j}(\theta)=c_j(\theta)=\frac{\cos j\theta}{\sqrt{\alpha(j)}}$ and $\epsilon_{2j+1}(\theta)=s_j(\theta)=\frac{\sin j\theta}{\sqrt{\alpha(j)}}$. If $\Gamma=\Gamma_{\rm LC}$, Γ_A or any Γ as above, we denote $\Gamma^{\epsilon_j}_{\epsilon_k\epsilon_r}=\Gamma(\epsilon_r,\epsilon_k,\epsilon_j)=(\Gamma(\epsilon_k)\epsilon_r\mid\epsilon_j)$. Let δ^p_r be the Kronecker symbol. The symbol

$$A_{jkp}=\delta^{k+p}_j\frac{(j+k)\sqrt{\alpha(p)}}{\sqrt{\alpha(j)}\sqrt{\alpha(k)}}\quad\text{for } k,j,p\geqslant 2,\tag{20}$$

is convenient to calculate in a systematic way the composition of Γ's.

$$\begin{aligned}\frac{1}{2}[A_{jkp}+A_{pkj}]&=\frac{1}{2}\big[\big([c_p,s_j]\mid c_k\big)+\big([c_p,c_k]\mid s_j\big)-\big([s_j,c_k]\mid c_p\big)\big]=(\Gamma_{\rm LC})^{c_p}_{c_ks_j}=-(\Gamma_{\rm LC})^{s_p}_{c_kc_j},\\\frac{1}{2}[A_{jkp}-A_{pkj}]&=\frac{1}{2}\big[-\big([c_p,c_j]\mid s_k\big)-\big([c_p,s_k]\mid c_j\big)+\big([c_j,s_k]\mid c_p\big)\big]=-(\Gamma_{\rm LC})^{c_p}_{s_kc_j}\\&=\frac{1}{2}\big[\big([s_j,s_p]\mid s_k\big)+\big([s_j,s_k]\mid s_p\big)-\big([s_p,s_k]\mid s_j\big)\big]=-(\Gamma_{\rm LC})^{s_p}_{s_ks_j}.\end{aligned}$$

The following operators E_j are diagonal in the basis $\{c_m,s_m\}_{k\geqslant 2}$ and the series $\sum_{j\geqslant 2}E_j$ converge,

$$\begin{aligned}&\big((\Gamma_{\rm LC}(c_j)^2+(\Gamma_{\rm LC})(s_j)^2)\big)c_m=-\frac{1}{2}\big[A^2_{mjr}+A^2_{rjm}\big]c_m,\\&\big[(\Gamma_{\rm LC}+\Gamma_1)(c_j)\Gamma_{\rm LC}(c_j)+(\Gamma_{\rm LC}+\Gamma_1)(s_j)\Gamma_{\rm LC}(s_j)\big]c_m\\&\quad=-\frac{1}{2}A_{mjr}A_{mrj}c_m=-\frac{(2m-j)(m+j)}{2\alpha(m)}1_{m\geqslant j+2}c_m,\end{aligned}$$

$$\begin{aligned}
&\big[\Gamma_4(c_j)\Gamma_1(c_j)+\Gamma_4(s_j)\Gamma_1(s_j)\big]c_m\\
&\quad=\left[\frac{1}{2}\big(A_{jmr}A_{jrm}-A_{rmj}A_{rjm}+A_{rjm}^2\big)+\frac{1}{2}\big(A_{mjr}^2+A_{mjr}A_{mrj}\big)\right]c_m,\\
&\big[\Gamma_4(c_j)\Gamma_2(c_j)+\Gamma_4(s_j)\Gamma_2(s_j)\big]c_m=\left[\frac{1}{2}\big(A_{rjm}^2-A_{jrm}^2\big)+\frac{1}{2}\big(A_{mjr}^2-A_{mrj}^2\big)\right]c_m,\\
&\big[\Gamma_1(c_j)\Gamma_2(c_j)+\Gamma_1(s_j)\Gamma_2(s_j)\big]c_m=\left[-\frac{1}{2}\big[A_{jrm}^2+A_{rjm}^2\big]-\frac{1}{2}(A_{mrj}-A_{mjr})^2\right]c_m,\\
&\big[\Gamma_{\mathrm{LC}}(c_j)\Gamma_2(c_j)+\Gamma_{\mathrm{LC}}(s_j)\Gamma_2(s_j)\big]c_m=\left[\frac{1}{2}A_{rjm}^2+\frac{1}{2}A_{mjr}(A_{mjr}-A_{mrj})\right]c_m.
\end{aligned}$$

The operator Λ defined by (5) is a bounded operator with the metric $\alpha(k)=k^3-k$,

$$\begin{aligned}
2\Lambda c_m&=-\sum_{j\geqslant 2}A_{mjr}A_{mrj}c_m=-\sum_{j\geqslant 2}\frac{(2m-j)(m+j)}{\alpha(m)}1_{m\geqslant j+2}c_m\\
&=-\frac{13}{6}+\text{a finite number of terms in }\frac{1}{m}.
\end{aligned}\tag{21}$$

Λ is related to the *tangent processes* introduced in the works [7,9]. Among all the Driver's connections, see [8], Γ_{LC} and Γ_A are the only ones for which $\sum_{j\geqslant 2}\Gamma^2(c_j)+\Gamma^2(s_j)$ converge. The series $\sum_{j\geqslant 2}\Gamma^2(c_j)+\Gamma^2(s_j)$ converge for Γ_{LC} and Γ_4, it diverge for $\Gamma=\Gamma_3$, Γ_5 and Γ_1, Γ_2.

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