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Mathematical Problems in Mechanics/Differential Geometry

Continuity in H^1-norms of surfaces in terms of the L^1-norms of their fundamental forms

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Abstract

The main purpose of this Note is to show how a 'nonlinear Korn's inequality on a surface' can be established. This inequality implies in particular the following interesting *per se* sequential continuity property for a sequence of surfaces. Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{R}^3$ be a smooth immersion, and let $\boldsymbol{\theta}^k:\overline{\omega}\to\mathbb{R}^3$, $k\geqslant 1$, be mappings with the following properties: They belong to the space $\boldsymbol{H}^1(\omega)$; the vector fields normal to the surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, are well defined a.e. in ω and they also belong to the space $\boldsymbol{H}^1(\omega)$; the principal radii of curvature of the surfaces $\boldsymbol{\theta}^k(\omega)$ stay uniformly away from zero; and finally, the three fundamental forms of the surfaces $\boldsymbol{\theta}^k(\omega)$ converge in $\boldsymbol{L}^1(\omega)$ toward the three fundamental forms of the surface $\boldsymbol{\theta}(\omega)$ as $k\to\infty$. Then, up to proper isometries of $\mathbb{R}^3$, the surfaces $\boldsymbol{\theta}^k(\omega)$ converge in $\boldsymbol{H}^1(\omega)$ toward the surface $\boldsymbol{\theta}(\omega)$ as $k\to\infty$. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Continuité en norme $\boldsymbol{H^1}$ de surfaces en terme des normes $\boldsymbol{L^1}$ de leurs formes fondamentales. L'objectif principal de cette Note est de montrer comment on peut établir une « inégalité de Korn non linéaire sur une surface ». Cette inégalité implique en particulier la propriété de continuité séquentielle suivante, intéressante par elle-même. Soit ω un domaine de $\mathbb{R}^2$, soit $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{R}^3$ une immersion régulière, et soit $\boldsymbol{\theta}^k:\overline{\omega}\to\mathbb{R}^3$, $k\geqslant 1$, des applications ayant les propriétés suivantes : Elles appartiennent à l'espace $\boldsymbol{H}^1(\omega)$; les champs de vecteurs normaux aux surfaces $\boldsymbol{\theta}^k(\omega)$, $k\geqslant 1$, sont définis presque partout dans ω et appartiennent aussi à l'espace $\boldsymbol{H}^1(\omega)$; les modules des rayons de courbure principaux des surfaces $\boldsymbol{\theta}^k(\omega)$ sont uniformément minorés par une constante strictement positive ; finalement, les trois formes fondamentales des surfaces $\boldsymbol{\theta}^k(\omega)$ convergent dans $\boldsymbol{L}^1(\omega)$ vers les trois formes fondamentales de la surface $\boldsymbol{\theta}(\omega)$ lorsque $k\to\infty$. Alors, à des isométries propres de $\mathbb{R}^3$ près, les surfaces $\boldsymbol{\theta}^k(\omega)$ convergent dans $\boldsymbol{H}^1(\omega)$ vers la surface $\boldsymbol{\theta}(\omega)$ lorsque $k\to\infty$. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

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1. Notations and other preliminaries

The symbols $\mathbb{M}^n$, $\mathbb{S}^n$, and $\mathbb{O}^n_+$ respectively designate the sets of all real matrices of order n, of all real symmetric matrices of order n, and of all real orthogonal matrices $\boldsymbol{R}$ of order n with $\det \boldsymbol{R} = 1$. The Euclidean norm of a vector $\boldsymbol{b} \in \mathbb{R}^n$ is denoted $|\boldsymbol{b}|$ and $|\boldsymbol{A}| := \sup_{|\boldsymbol{b}|=1} |\boldsymbol{A}\boldsymbol{b}|$ denotes the spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^n$.

Let U be an open subset in $\mathbb{R}^n$. Given any smooth enough mapping $\boldsymbol{\chi} : U \to \mathbb{R}^n$, we let $\boldsymbol{\nabla}\boldsymbol{\chi}(x) \in \mathbb{M}^n$ denote the gradient matrix of the mapping $\boldsymbol{\chi}$ at $x \in U$ and we let $\partial_i \boldsymbol{\chi}(x)$ denote the ith column of the matrix $\boldsymbol{\nabla}\boldsymbol{\chi}(x)$. Given any mapping $\boldsymbol{F} \in L^1(U; \mathbb{S}^n)$, we let

$$\|\boldsymbol{F}\|_{L^1(U;\mathbb{S}^n)} := \int_U \bigl|\boldsymbol{F}(x)\bigr| \,\mathrm{d}x,$$

and, given any mapping $\boldsymbol{\chi} \in H^1(U; \mathbb{R}^n)$, we let

$$\|\boldsymbol{\chi}\|_{H^1(U;\mathbb{R}^n)} := \left\{ \int_U \left(\bigl|\boldsymbol{\chi}(x)\bigr|^2 + \sum_{i=1}^n \bigl|\partial_i \boldsymbol{\chi}(x)\bigr|^2 \right) \mathrm{d}x \right\}^{1/2}.$$

A *domain* U in $\mathbb{R}^n$ is an open and bounded subset of $\mathbb{R}^n$ with a boundary that is Lipschitz-continuous in the sense of Adams [1] or Nečas [10], the set U being locally on the same side of its boundary. If U is a domain in $\mathbb{R}^n$, the space $\mathcal{C}^1(\overline{U}; \mathbb{R}^m)$ consists of all vector-valued mappings $\boldsymbol{\chi} \in \mathcal{C}^1(U; \mathbb{R}^m)$ that, together with all their partial derivatives of the first order, possess continuous extensions to the closure $\overline{U}$ of U. The space $\mathcal{C}^1(\overline{U}; \mathbb{R}^m)$ thus also consists of restrictions to $\overline{U}$ of all mappings in the space $\mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^m)$ (for a proof, see, e.g., [13] or [7]).

Latin indices and exponents henceforth range in the set $\{1, 2, 3\}$ save when they are used for indexing sequences, Greek indices and exponents range in the set $\{1, 2\}$, and the summation convention is used in conjunction with these rules.

The notations $(a_{\alpha\beta})$, $(a^{\alpha\beta})$, (b^β_α), and (g_{ij}) respectively designate matrices in $\mathbb{M}^2$ and $\mathbb{M}^3$ with components $a_{\alpha\beta}$, $a^{\alpha\beta}$, b^β_α, and g_{ij}, the index or exponent α and the index i designating here the row index.

Complete proofs of the results announced in this Note are found in [3].

2. A nonlinear Korn inequality on a surface

Our main result is a *nonlinear Korn inequality on a surface* (Theorem 2.4), the proof of which relies on several preliminaries, a crucial one being the following *nonlinear Korn inequality on an open subset in* $\mathbb{R}^n$ recently established by Ciarlet and Mardare [6]. Its long, and sometimes technical, proof hinges in particular on a fundamental 'geometric rigidity lemma' due to Friesecke et al. [9] and on a general methodology reminiscent to that used in Ciarlet and Laurent [4]. See also Reshetnyak [12] for related results.

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^n$. Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla}\boldsymbol{\Theta} > 0$ *in $\overline{\Omega}$, there exists a constant $C(\boldsymbol{\Theta})$ with the following property*: *Given any mapping $\widetilde{\boldsymbol{\Theta}} \in H^1(\Omega; \mathbb{R}^n)$ satisfying* $\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} > 0$ *a.e. in Ω, there exist a vector $\boldsymbol{b} = \boldsymbol{b}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{R}^n$ and a matrix $\boldsymbol{R} = \boldsymbol{R}(\widetilde{\boldsymbol{\Theta}}, \boldsymbol{\Theta}) \in \mathbb{O}^n_+$ such that*

$$\bigl\|(\boldsymbol{b} + \boldsymbol{R}\widetilde{\boldsymbol{\Theta}}) - \boldsymbol{\Theta}\bigr\|_{H^1(\Omega;\mathbb{R}^n)} \leqslant C(\boldsymbol{\Theta}) \bigl\|\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^{\mathrm{T}} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}} - \boldsymbol{\nabla}\boldsymbol{\Theta}^{\mathrm{T}} \boldsymbol{\nabla}\boldsymbol{\Theta}\bigr\|^{1/2}_{L^1(\Omega;\mathbb{S}^n)}.$$

The next two lemmas show that some classical definitions and properties pertaining to surfaces in $\mathbb{R}^3$ still hold under less stringent regularity assumptions than the usual ones (these definitions and properties are traditionally given and established under the assumptions that the immersions denoted $\boldsymbol{\theta}$ in Lemma 2.2 and $\tilde{\boldsymbol{\theta}}$ in Lemma 2.3 belong to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$). For this reason, we shall continue to use the classical terminology, e.g., normal vector field (for $\boldsymbol{a}_3$ or $\tilde{\boldsymbol{a}}_3$), or first, second, and third fundamental forms (for $(a_{\alpha\beta})$ or $(\tilde{a}_{\alpha\beta})$, $(b_{\alpha\beta})$ or $(\tilde{b}_{\alpha\beta})$, and $(c_{\alpha\beta})$ or $(\tilde{c}_{\alpha\beta})$), etc. If $y = (y_\alpha)$ designates the generic point in a domain ω in $\mathbb{R}^2$, we let $\partial_\alpha := \partial/\partial y_\alpha$.

Lemma 2.2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ be an immersion such that $\boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$, where $\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta}$. Then the functions*

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad b_{\alpha\beta} := -\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta, \quad b_\alpha^\sigma := a^{\beta\sigma} b_{\alpha\beta}, \quad c_{\alpha\beta} := \partial_\alpha \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_3,$$

where $(a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}$, belong to the space $\mathcal{C}^0(\overline{\omega})$, and $b_{\alpha\beta} = b_{\beta\alpha}$. Define the mapping $\boldsymbol{\Theta} : \overline{\omega} \times \mathbb{R} \to \mathbb{R}^3$ by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \quad \text{for all } (y, x_3) \in \overline{\omega} \times \mathbb{R}.$$

Then $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\omega} \times \mathbb{R}; \mathbb{R}^3)$. Furthermore,

$$\det \boldsymbol{\nabla\Theta}(y, x_3) = \sqrt{a(y)}\left\{1 - 2H(y)x_3 + K(y)x_3^2\right\} \quad \text{for all } (y, x_3) \in \overline{\omega} \times \mathbb{R},$$

where the functions

$$a := \det(a_{\alpha\beta}) = |\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|^2, \quad H := \frac{1}{2}\left(b_1^1 + b_2^2\right), \quad K := b_1^1 b_2^2 - b_1^2 b_2^1$$

belong to the space $\mathcal{C}^0(\overline{\omega})$. Finally, let

$$(g_{ij}) := \boldsymbol{\nabla\Theta}^{\mathrm{T}} \boldsymbol{\nabla\Theta}.$$

Then the functions $g_{ij} = g_{ji}$ belong to the space $\mathcal{C}^0(\overline{\omega} \times \mathbb{R})$ and they are given by

$$g_{\alpha\beta}(y, x_3) = a_{\alpha\beta}(y) - 2x_3 b_{\alpha\beta}(y) + x_3^2 c_{\alpha\beta}(y) \quad \text{and} \quad g_{i3}(y, x_3) = \delta_{i3}$$

for all $(y, x_3) \in \overline{\omega} \times \mathbb{R}$.

Sketch of proof. Since the symmetric matrices $(a_{\alpha\beta}(y))$ are positive-definite at all points $y \in \overline{\omega}$, the inverse matrices $(a^{\alpha\beta}(y))$ are well defined and also positive-definite at all points $y \in \overline{\omega}$, and the functions $a^{\alpha\beta}$ belong to the space $\mathcal{C}^0(\overline{\omega})$. Therefore the functions b_α^σ are well-defined and they also belong to the space $\mathcal{C}^0(\overline{\omega})$.

The symmetry $b_{\alpha\beta} = b_{\beta\alpha}$ is clear if $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ since $b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta$ in this case. As shown in the proof of Theorem 3 of Ciarlet and Mardare [5], this symmetry still holds under the weaker assumptions of Lemma 2.2.

Thanks to the relations $\partial_\alpha(\boldsymbol{a}_3 \cdot \boldsymbol{a}_3) = 0$, the classical formulas of Weingarten, viz.,

$$\partial_\alpha \boldsymbol{a}_3 = -b_\alpha^\sigma \boldsymbol{a}_\sigma,$$

still hold under the present assumptions. The expressions giving the functions $\det \boldsymbol{\nabla\Theta}$ and g_{ij} then follow from this observation. □

Lemma 2.3. *Let ω be a domain in $\mathbb{R}^2$ and let there be given a mapping $\tilde{\boldsymbol{\theta}} \in H^1(\omega; \mathbb{R}^3)$ such that $\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2 \neq \boldsymbol{0}$ a.e. in ω, where $\tilde{\boldsymbol{a}}_\alpha := \partial_\alpha \tilde{\boldsymbol{\theta}}$, and such that*

$$\tilde{\boldsymbol{a}}_3 := \frac{\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2}{|\tilde{\boldsymbol{a}}_1 \wedge \tilde{\boldsymbol{a}}_2|} \in H^1(\omega; \mathbb{R}^3).$$

Then the functions

$$\tilde{a}_{\alpha\beta} := \tilde{\boldsymbol{a}}_\alpha \cdot \tilde{\boldsymbol{a}}_\beta, \quad \tilde{b}_{\alpha\beta} := -\partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \tilde{\boldsymbol{a}}_\beta, \quad \tilde{c}_{\alpha\beta} := \partial_\alpha \tilde{\boldsymbol{a}}_3 \cdot \partial_\beta \tilde{\boldsymbol{a}}_3$$

are well defined a.e. in ω, they belong to the space $L^1(\omega)$, and $\tilde{b}_{\alpha\beta} = \tilde{b}_{\beta\alpha}$. Define the mapping $\widetilde{\boldsymbol{\Theta}} : \omega \times \mathbb{R} \to \mathbb{R}^3$ by

$$\widetilde{\boldsymbol{\Theta}}(y, x_3) := \tilde{\boldsymbol{\theta}}(y) + x_3 \tilde{\boldsymbol{a}}_3(y) \quad \text{for almost all } (y, x_3) \in \omega \times \mathbb{R}.$$

Then $\widetilde{\boldsymbol{\Theta}} \in H^1(\omega \times]-\delta, \delta[; \mathbb{R}^3)$ for any $\delta > 0$. Furthermore,

$$\det \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(y, x_3) = \sqrt{\tilde{a}(y)}\left\{1 - 2\widetilde{H}(y)x_3 + \widetilde{K}(y)x_3^2\right\} \quad \text{for almost all } (y, x_3) \in \omega \times \mathbb{R},$$

where

$$\tilde a := \det(\tilde a_{\alpha\beta}) = |\tilde{\boldsymbol a}_1 \wedge \tilde{\boldsymbol a}_2|^2, \quad \widetilde H := \frac{1}{2}\big(\tilde b_1^1 + \tilde b_2^2\big), \quad \widetilde K := \tilde b_1^1 \tilde b_2^2 - \tilde b_1^2 \tilde b_2^1, \quad \tilde b_\alpha^\sigma := \tilde a^{\beta\sigma}\tilde b_{\alpha\beta},$$

and $(\tilde a^{\alpha\beta}) := (\tilde a_{\alpha\beta})^{-1}$. *Finally, let*

$$(\tilde g_{ij}) := \nabla\widetilde{\boldsymbol\Theta}^{\mathrm T}\nabla\widetilde{\boldsymbol\Theta} \quad \text{a.e. in } \omega\times\mathbb R.$$

Then the functions $\tilde g_{ij} = \tilde g_{ji}$ *belong to the space* $L^1(\omega\times\,]-\delta,\delta[)$ *for any* $\delta > 0$ *and they are given by*

$$\tilde g_{\alpha\beta}(y,x_3) = \tilde a_{\alpha\beta}(y) - 2x_3\tilde b_{\alpha\beta}(y) + x_3^2\tilde c_{\alpha\beta}(y) \quad \text{and} \quad \tilde g_{i3}(y,x_3) = \delta_{i3}$$

for almost all $(y,x_3)\in\omega\times\mathbb R$.

Sketch of proof. The proof is analoguous to that of Lemma 2.2. The symmetry $\tilde b_{\alpha\beta} = \tilde b_{\beta\alpha}$ again follow from Theorem 3 of [5]. Note that, although the functions $\tilde a$, $\widetilde H$, $\widetilde K$ and $\tilde b_\alpha^\sigma$ are well defined a.e. in ω under the assumptions of Lemma 2.3, they do not necessarily belong to the space $L^1(\omega)$. □

We now state the announced *nonlinear Korn inequality on a surface*. The notations are the same as those in Lemmas 2.2 and 2.3.

Theorem 2.4. *Let there be given a domain* ω *in* $\mathbb R^2$, *an immersion* $\boldsymbol\theta\in\mathcal C^1(\overline\omega;\mathbb R^3)$ *such that* $\boldsymbol a_3\in\mathcal C^1(\overline\omega;\mathbb R^3)$, *and* $\varepsilon > 0$.

Then there exists a constant $c(\boldsymbol\theta,\varepsilon)$ *with the following property*: *Given any mapping* $\tilde{\boldsymbol\theta}\in H^1(\omega;\mathbb R^3)$ *such that* $\tilde{\boldsymbol a}_1\wedge\tilde{\boldsymbol a}_2\neq 0$ *a.e. in* ω, $\tilde{\boldsymbol a}_3\in H^1(\omega;\mathbb R^3)$, *and*

$$|\widetilde H| \leqslant \frac{1}{\varepsilon} \quad \text{and} \quad \widetilde K \geqslant -\frac{1}{\varepsilon^2} \quad \text{a.e. in } \omega,$$

there exist a vector $\boldsymbol b := \boldsymbol b(\boldsymbol\theta,\tilde{\boldsymbol\theta},\varepsilon)\in\mathbb R^3$ *and a matrix* $\boldsymbol R = \boldsymbol R(\boldsymbol\theta,\tilde{\boldsymbol\theta},\varepsilon)\in\mathbb O^3_+$ *such that*

$$\begin{aligned}&\big\|(\boldsymbol b + \boldsymbol R\tilde{\boldsymbol\theta}) - \boldsymbol\theta\big\|_{H^1(\omega;\mathbb R^3)} + \varepsilon\|\boldsymbol R\tilde{\boldsymbol a}_3 - \boldsymbol a_3\|_{H^1(\omega;\mathbb R^3)}\\ &\quad\leqslant c(\boldsymbol\theta,\varepsilon)\big\{\big\|(\tilde a_{\alpha\beta} - a_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb S^2)} + \varepsilon^{1/2}\big\|(\tilde b_{\alpha\beta} - b_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb S^2)} + \varepsilon\big\|(\tilde c_{\alpha\beta} - c_{\alpha\beta})\big\|^{1/2}_{L^1(\omega;\mathbb S^2)}\big\}.\end{aligned}$$

Sketch of proof. Without loss of generality, we assume that $\varepsilon\leqslant 1$. Let the mappings $\boldsymbol\Theta:\overline\omega\times\mathbb R\to\mathbb R^3$ and $\widetilde{\boldsymbol\Theta}:\omega\times\mathbb R\to\mathbb R^3$ be constructed as in Lemmas 2.2 and 2.3 from the mappings $\boldsymbol\theta:\overline\omega\to\mathbb R^3$ and $\tilde{\boldsymbol\theta}:\omega\to\mathbb R^3$ appearing in Theorem 2.4. Then there exists a constant $\delta(\boldsymbol\theta) > 0$ such that

$$\det\nabla\boldsymbol\Theta > 0 \quad \text{in } \overline\Omega \quad \text{and} \quad \det\nabla\widetilde{\boldsymbol\Theta} > 0 \quad \text{a.e. in } \Omega,$$

where $\Omega = \Omega(\boldsymbol\theta,\varepsilon) := \omega\times\,]-\delta(\boldsymbol\theta)\varepsilon,\delta(\boldsymbol\theta)\varepsilon[$.

Theorem 2.1 then shows that there exists a constant $c_0(\boldsymbol\theta,\varepsilon)$ with the following property: Given any $\varepsilon > 0$ and given any mappings $\boldsymbol\theta$ and $\tilde{\boldsymbol\theta}$ satisfying the assumptions of Theorem 2.4, there exist a vector $\boldsymbol b := \boldsymbol b(\boldsymbol\theta,\tilde{\boldsymbol\theta},\varepsilon)\in\mathbb R^3$ and a matrix $\boldsymbol R = \boldsymbol R(\boldsymbol\theta,\tilde{\boldsymbol\theta},\varepsilon)\in\mathbb O^3_+$ such that

$$\big\|(\boldsymbol b + \boldsymbol R\widetilde{\boldsymbol\Theta}) - \boldsymbol\Theta\big\|_{H^1(\Omega;\mathbb R^3)} \leqslant c_0(\boldsymbol\theta,\varepsilon)\big\|(\tilde g_{ij} - g_{ij})\big\|^{1/2}_{L^1(\Omega;\mathbb S^3)}.$$

The rest of the proof consists in showing that there exists constants $c_1(\boldsymbol\theta)$ and $c_2(\boldsymbol\theta)$ such that

$$\big\|(\boldsymbol b + \boldsymbol R\widetilde{\boldsymbol\Theta}) - \boldsymbol\Theta\big\|_{H^1(\Omega;\mathbb R^3)} \geqslant c_1(\boldsymbol\theta)\varepsilon^{1/2}\big\{\big\|(\boldsymbol b + \boldsymbol R\tilde{\boldsymbol\theta}) - \boldsymbol\theta\big\|_{H^1(\omega;\mathbb R^3)} + \varepsilon\|\boldsymbol R\tilde{\boldsymbol a}_3 - \boldsymbol a_3\|_{H^1(\omega;\mathbb R^3)}\big\},$$

and

$$\begin{aligned}&\|(\tilde g_{ij}-g_{ij})\|^{1/2}_{L^1(\Omega;\mathbb{S}^3)}\\&\quad\leqslant c_2(\boldsymbol\theta)\varepsilon^{1/2}\big\{\|(\tilde a_{\alpha\beta}-a_{\alpha\beta})\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon^{1/2}\|(\tilde b_{\alpha\beta}-b_{\alpha\beta})\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}+\varepsilon\|(\tilde c_{\alpha\beta}-c_{\alpha\beta})\|^{1/2}_{L^1(\omega;\mathbb{S}^2)}\big\}.\end{aligned}$$

The announced inequality then follows with $c(\boldsymbol\theta,\varepsilon):=c_0(\boldsymbol\theta,\varepsilon)c_1(\boldsymbol\theta)^{-1}c_2(\boldsymbol\theta)$. □

3. Commentary

If a mapping $\tilde{\boldsymbol\theta}:\omega\to\mathbb{R}^3$ is a smooth immersion, the associated functions $\widetilde H$ and $\widetilde K$ simply represent the *mean*, and *Gaussian*, *curvatures* of the surface $\tilde{\boldsymbol\theta}(\omega)$. It is well known that these functions are also given by $\widetilde H=\frac12(\frac{1}{\widetilde R_1}+\frac{1}{\widetilde R_2})$ and $\widetilde K=\frac{1}{\widetilde R_1\widetilde R_2}$, where $\widetilde R_\alpha$ are the *principal radii of curvature* along the surface $\tilde{\boldsymbol\theta}(\omega)$ (with the usual convention that $|R_\alpha(y)|$ may take the value $+\infty$ at some points $y\in\omega$).

It is then easily seen that the assumptions $|\widetilde H|\leqslant\frac{1}{\varepsilon}$ and $\widetilde K\geqslant-\frac{1}{\varepsilon^2}$ in ω made in Theorem 2.4 imply that $|\widetilde R_\alpha|\geqslant c\varepsilon$ in ω and that, conversely, $|\widetilde R_\alpha|\geqslant\varepsilon$ in ω implies that $|\widetilde H|\leqslant\frac{d}{\varepsilon}$ and $\widetilde K\geqslant-\frac{d}{\varepsilon^2}$ in ω, for some ad hoc numerical constants c and d. Hence the assumptions made on the mappings $\tilde{\boldsymbol\theta}$ in Theorem 2.4 have a very simple geometric interpretation: they mean that *the principal radii of curvature of all the 'admissible' surfaces $\tilde{\boldsymbol\theta}(\omega)$ must stay uniformly away from zero*. Naturally, such principal radii of curvature are possibly understood only in a generalized sense, viz., as the inverses of the eigenvalues of the associated matrix $(\tilde b_\alpha^\beta)$.

Let there be given a mapping $\tilde{\boldsymbol\theta}\in H^1(\omega;\mathbb{R}^3)$ such that $\tilde{\boldsymbol a}_1\wedge\tilde{\boldsymbol a}_2\neq\mathbf 0$ a.e. in ω and $\tilde{\boldsymbol a}_3\in H^1(\omega;\mathbb{R}^3)$. Then a mapping $\hat{\boldsymbol\theta}:\omega\to\mathbb{R}^3$ is said to be *properly isometrically equivalent* to the mapping $\tilde{\boldsymbol\theta}$ if there exist a vector $\boldsymbol b\in\mathbb{R}^3$ and a matrix $\boldsymbol R\in\mathbb{O}^3_+$ such that $\hat{\boldsymbol\theta}=\boldsymbol b+\boldsymbol R\tilde{\boldsymbol\theta}$. If this is the case, then $\hat{\boldsymbol\theta}\in H^1(\omega;\mathbb{R}^3)$, $\hat{\boldsymbol a}_1\wedge\hat{\boldsymbol a}_2\neq\mathbf 0$ a.e. in ω, and $\hat{\boldsymbol a}_3\in H^1(\omega;\mathbb{R}^3)$ (with self-explanatory notations), and the two surfaces $\tilde{\boldsymbol\theta}(\omega)$ and $\hat{\boldsymbol\theta}(\omega)$ share the same three fundamental forms in the space $L^1(\omega;\mathbb{S}^2)$.

One application of the key inequality of Theorem 2.4 is then the following result of *sequential continuity for surfaces*: Let $\boldsymbol\theta^k\in H^1(\omega;\mathbb{R}^3)$, $k\geqslant1$, be mappings with the following properties: The vector fields normal to the surfaces $\boldsymbol\theta^k(\omega)$ are well defined a.e. in ω and they also belong to the space $H^1(\omega;\mathbb{R}^3)$, there exists a constant $\varepsilon>0$ such that the principal radii of curvatures R^k_α of the surfaces $\boldsymbol\theta^k(\omega)$ satisfy $|R^k_\alpha|\geqslant\varepsilon>0$ a.e. in ω for all $k\geqslant1$, and finally,

$$\big(a^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(a_{\alpha\beta}),\quad \big(b^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(b_{\alpha\beta}),\quad \big(c^k_{\alpha\beta}\big)\underset{k\to\infty}{\longrightarrow}(c_{\alpha\beta})\quad\text{in } L^1(\omega;\mathbb{S}^2),$$

where $(a_{\alpha\beta})$, $(b_{\alpha\beta})$, $(c_{\alpha\beta})$ are the three fundamental forms of a surface $\boldsymbol\theta(\omega)$, where $\boldsymbol\theta\in\mathcal C^1(\overline\omega;\mathbb{R}^3)$ is an immersion satisfying $\boldsymbol a_3\in\mathcal C^1(\overline\omega;\mathbb{R}^3)$. Then *there exist mappings $\hat{\boldsymbol\theta}^k$ that are properly isometrically equivalent to the mappings $\boldsymbol\theta^k$, $k\geqslant1$, such that*

$$\hat{\boldsymbol\theta}^k\underset{k\to\infty}{\longrightarrow}\boldsymbol\theta\quad\text{and}\quad\hat{\boldsymbol a}^k_3\underset{k\to\infty}{\longrightarrow}\boldsymbol a_3\quad\text{in } H^1(\omega;\mathbb{R}^3).$$

Such a sequential continuity property generalizes that previously obtained by Ciarlet [2] and by Ciarlet and Mardare [8] and Szopos [11], for the topologies of the spaces $\mathcal C^m(\omega)$, and $\mathcal C^m(\overline\omega)$, respectively.

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