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Algebraic Geometry

On the Łojasiewicz numbers, II

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Abstract

For every holomorphic function in two complex variables with an isolated critical point at the origin we consider the Łojasiewicz exponent $\mathcal{L}_0(f)$ defined to be the smallest $\theta > 0$ such that $|\mathrm{grad}\, f(z)| \geqslant c|z|^\theta$ near $0 \in \mathbb{C}^2$ for some $c > 0$. The numbers $\mathcal{L}_0(f)$ are rational. In this Note we discuss the interplay between arithmetical properties of the rationals $\mathcal{L}_0(f)$ and topological properties of plane curve singularities $f = 0$. ***To cite this article: E. García Barroso et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sur les nombres de Łojasiewicz, II. Pour toute fonction holomorphe f de deux variables complexes ayant un point critique isolé à l'origine nous considérons l'exposant de Łojasiewicz $\mathcal{L}_0(f)$ égal, par definition, au plus petit nombre $\theta > 0$ tel que $|\mathrm{grad}\, f(z)| \geqslant c|z|^\theta$ dans un voisinage de $0 \in \mathbb{C}^2$. Dans cette Note nous étudions le rapport entre des propriétés arithmétiques de l'exposant $\mathcal{L}_0(f)$ et des propriétés topologiques de la singularité plane $f = 0$. ***Pour citer cet article : E. García Barroso et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

Let f be a holomorphic function defined near $0 \in \mathbb{C}^2$, $f(0) = 0$, with an isolated critical point at the origin and let $(C, 0)$ be the germ of a singular plane curve with local equation $f = 0$. Set $\mathrm{grad}\, f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. The *Łojasiewicz exponent* $\mathcal{L}_0(f)$ *of* f *at* 0 is defined to be the smallest $\theta > 0$ such that

$$\bigl|\mathrm{grad}\, f(z)\bigr| \geqslant c|z|^\theta \quad \text{in a neighbourhood of } 0 \in \mathbb{C}^2 \text{ with a constant } c > 0. \tag{1}$$

Teissier proved (see [6], p. 275) that the Łojasiewicz exponent $\mathcal{L}_0(f)$ depends only on the topological type of the germ $(C, 0)$; more specifically $\mathcal{L}_0(f) + 1$ is the maximal polar invariant of $(C, 0)$. In particular $\mathcal{L}_0(f)$ is a rational number. In this Note we will consider the problem as to which rational numbers are Łojasiewicz exponents of plane curve singularities. Such numbers will be called *Łojasiewicz numbers*.

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In [5], p. 359, it was proved that every Łojasiewicz number appears in the sequence $1, 2, 3, 3\frac{1}{2}, 4, 4\frac{1}{3}, 4\frac{1}{2}, 4\frac{2}{3}, 5, \ldots,$ which terms greater than 1 are rationals of the form

$$N + \frac{b}{a} \quad \text{where } a, b, N \text{ are integers such that } 0 \leqslant b < a < N. \tag{2}$$

In [1], Theorem 1.3, the authors proved that numbers of (2) for which $a = N - 1$, $b > 1$ and $\mathrm{GCD}(a, b) = 1$ are not Łojasiewicz numbers.

In this Note we continue the study of the Łojasiewicz numbers. First we prove that all numbers of (2) for which $a + b \leqslant N$ and the number 1 are the Łojasiewicz exponents of nondegenerate, in Kouchnirenko's sense [2], plane curve singularities (Theorem 2.1). We call them *regular Łojasiewicz numbers*. All remaining Łojasiewicz numbers will be called *nonregular*. It turns out that the assumption $\mathcal{L}_0(f)$ is nonregular imposes strong restrictions on the singularity $f = 0$, expressed in terms of topological invariants of the singularity (Theorem 2.2). For example, all singularities with the Milnor number $\mu \leqslant 100$ have regular Łojasiewicz exponents. On the other hand we present a result (Theorem 2.3) on the existence of singularities with given Łojasiewicz numbers. It enables us to construct an infinite sequence of nonregular Łojasiewicz numbers.

Using the above quoted results we get Theorem 3.1 which gives information on the nonregular Łojasiewicz numbers in terms of characteristic sequences. Theorem 3.1 allows us to characterize the denominators of nonregular Łojasiewicz numbers (Theorems 3.2 and 3.3). From Theorem 3.2 we obtain easily the main result of [1].

In this note we follow the notations used by Zariski in [7] (see pp. 7–25). The sequence of positive integers $\beta_0, \ldots, \beta_g$ is called a *characteristic sequence* if

(i) $\beta_i < \beta_{i+1}$ for $i = 0, 1, \ldots, g - 1$ and
(ii) if $e_i = \mathrm{GCD}(\beta_0, \ldots, \beta_i)$ for $i = 0, 1, \ldots, g$, then $e_i > e_{i+1}$ for $i = 0, 1, \ldots, g - 1$ and $e_g = 1$.

For any characteristic sequence $\beta_0, \ldots, \beta_g$ we consider the *derived characteristic sequence* $\overline{\beta_0} = \beta_0$, $\overline{\beta_1} = \beta_1$, $\overline{\beta_i} = \beta_i + \frac{1}{e_{i-1}} \sum_{j=1}^{i-1} (e_{j-1} - e_j)\beta_j$ for $i = 2, \ldots, g$. The semigroup $\langle \overline{\beta_0}, \ldots, \overline{\beta_g} \rangle = \mathbb{N}\overline{\beta_0} + \cdots + \mathbb{N}\overline{\beta_g}$ plays an important role in the theory of branches (see [7]).

For every branch $(C, 0)$ we denote by $(\beta_0(C), \ldots, \beta_g(C))$ the *characteristic of* $(C, 0)$ (if $(C, 0)$ is nonsingular then $g = 0$ and $\beta_0(C) = 1$); $\beta_0(C) = m(C)$ is the multiplicity of $(C, 0)$. The characteristic $(\beta_0(C), \ldots, \beta_g(C))$ is a characteristic sequence. Every characteristic sequence is equal to the characteristic of a branch. The *semigroup* $\Gamma(C, 0)$ *of the branch* $(C, 0)$ is, by definition, generated by the intersection numbers $(C, D)_0$ where $(D, 0)$ runs over all plane curve germs such that $(C, 0) \not\subset (D, 0)$. It can be also described as $\langle \overline{\beta_0(C)}, \ldots, \overline{\beta_g(C)} \rangle$.

The proofs of the results announced in this Note will be published elsewhere.

2. Łojasiewicz numbers and singularities of plane curves

For every holomorphic function $f = \sum c_{\alpha\beta} x^\alpha y^\beta$ near $0 \in \mathbb{C}^2$ we consider the *Newton diagram* $\Delta(f)$ *of* f. Recall that $\Delta(f)$ is the convex hull of the set $\{(\alpha, \beta) \in \mathbb{N}^2\colon c_{\alpha\beta} \neq 0\} + \mathbb{R}^2_+$. For every compact face S of the boundary $\partial\Delta(f)$, we define the principal part $f_S = \sum_{(\alpha,\beta)\in S} c_{\alpha\beta} x^\alpha y^\beta$. The germ of f at $0 \in \mathbb{C}^2$ is *nondegenerate* (in Kouchnirenko's sense) if all principal parts f_S have no critical points in the set $(\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})$ (see [2]).

Our first result is an arithmetical characterization of the Łojasiewicz exponents of nondegenerate singularities.

Theorem 2.1. *A rational number* $\lambda > 0$ *is the Łojasiewicz number of a nondegenerate singularity if and only if* $\lambda = N + \frac{b}{a}$, *where* a, b, N *are integers such that* $0 \leqslant b < a$ *and* $a + b \leqslant N$.

The *only if* part follows from an explicit formula for the Łojasiewicz exponent of a nondegenerate singularity ([4], Theorem 1). We get the *if* part from

Example 1. Let N, a, b be integers such that $0 < b < a$, $a + b \leqslant N$ and $\mathrm{GCD}(a, b) = 1$. We put $f(x, y) = y^{N+2} + xy^{N+1} + x^{a+1}y^{N-a-b} + x^N$ if $a + b < N$ and $f(x, y) = y^{a+1} + yx^N$ if $a + b = N$. Then f is nondegenerate in Kouchnirenko's sense and $\mathcal{L}_0(f) = N + \frac{b}{a}$.

We have defined the regular Łojasiewicz numbers as the Łojasiewicz numbers $\mathcal{L}_0(f)$ for which

$$\mathcal{L}_0(f) = N + \frac{b}{a}, \quad 0 \leqslant b < a,\ a + b \leqslant N.$$

All remaining Łojasiewicz numbers we have called nonregular. Thus $\mathcal{L}_0(f)$ is nonregular if and only if

$$\mathcal{L}_0(f) = N + \frac{b}{a}, \quad 0 < b < a < N,\ \mathrm{GCD}(a, b) = 1,\ a + b > N.$$

According to Theorem 2.1 the regular Łojasiewicz numbers are the Łojasiewicz exponents of nondegenerate plane singularities. The next theorem gives necessary conditions for a plane curve singularity to have a nonregular Łojasiewicz exponent. Let $d(C, D) = (C, D)_0/(m(C)m(D))$.

Theorem 2.2. *Suppose that the Łojasiewicz number $\mathcal{L}_0(f)$ is nonregular. Then the germ $(C, 0)$ of a plane curve singularity with local equation $f = 0$ has at least two branches and there is a decomposition $(C, 0) = \bigcup_{i=1}^{r}(C_i, 0)$ into branches $(C_i, 0)$ such that the following conditions are fulfilled*:

(i) *The branch $(C_1, 0)$ is singular. If $(\beta_0, \ldots, \beta_g)$ is the characteristic of $(C_1, 0)$ then $g \geqslant 2$ and the sequence $(\frac{\beta_0}{e_{g-1}}, \ldots, \frac{\beta_{g-1}}{e_{g-1}})$ is the characteristic of $(C_2, 0)$. Moreover $(C_1, C_2)_0 = \overline{\beta_g}$.*
(ii) *For every $i \neq 1, 2$ $d(C_1, C_i) = d(C_2, C_i) < d(C_1, C_2)$.*
(iii) *$\mathcal{L}_0(f) + 1 = (e_{g-1}\overline{\beta_g} + \overline{\beta_g} + \delta)/\beta_0$ where $\delta = \sum_{i\neq 1,2}(C_1, C_i)_0$. Moreover $\delta \in \langle\overline{\beta_0}, \ldots, \overline{\beta_{g-1}}\rangle$.*

We complete Theorem 2.2 by Theorem 2.3 which gives a sufficient condition for a rational to be a Łojasiewicz number. The theorem enables us to construct nonregular Łojasiewicz numbers.

Theorem 2.3. *Let $(\beta_0, \ldots, \beta_g)$ be a characteristic sequence and let $\delta = \sum_{i=0}^{g-1} a_{i-1}\overline{\beta_i} \in \langle\overline{\beta_0}, \ldots, \overline{\beta_{g-1}}\rangle$ with $a_i = 0$ or $a_i = 1$ for $i \geqslant 0$. Then there exists a plane curve singularity $(C, 0)$ with a local equation $f = 0$ for which there is a decomposition into branches $(C, 0) = \bigcup_{i=1}^{r}(C_i, 0)$, $r > 1$, such that*

(i) *The branches $(C_1, 0)$ and $(C_2, 0)$ are of characteristic $(\beta_0, \ldots, \beta_g)$ and $(\frac{\beta_0}{e_{g-1}}, \ldots, \frac{\beta_{g-1}}{e_{g-1}})$, respectively. Moreover $(C_1, C_2)_0 = \overline{\beta_g}$.*
(ii) *For every $i \neq 1, 2$ $d(C_1, C_i) = d(C_2, C_i) < d(C_1, C_2)$.*
(iii) *$\mathcal{L}_0(f) + 1 = (e_{g-1}\overline{\beta_g} + \overline{\beta_g} + \delta)/\beta_0$ and $\delta = \sum_{i\neq 1,2}(C_1, C_i)_0$.*

Now we can construct a sequence of nonregular Łojasiewicz numbers.

Example 2. Let $p > 2$ be a prime number. Taking the characteristic sequence $(p^2, p^2 + p, p^2 + 2p - 1)$ and $\delta = 0$ we get by Theorem 2.3 that there is a two-branched singularity $f = 0$ such that $\mathcal{L}_0(f) = (p + 1)^2 - \frac{1}{p^2}$. It is easy to see that $\mathcal{L}_0(f)$ is a nonregular Łojasiewicz number.

3. Łojasiewicz numbers and characteristic sequences

Let $(\beta_0, \ldots, \beta_g)$ be a characteristic sequence. We say that a rational number λ *is associated with* $(\beta_0, \ldots, \beta_g)$ if there exists an integer $\delta \in \langle\overline{\beta_0}, \ldots, \overline{\beta_{g-1}}\rangle$ such that $\lambda + 1 = (e_{g-1}\overline{\beta_g} + \overline{\beta_g} + \delta)/\beta_0$. If we can choose $\delta = \sum_{i=0}^{g-1} a_{i-1}\overline{\beta_i}$ such that $a_i = 0$ or $a_i = 1$ for $i \geqslant 1$ then we say that λ *is strictly associated with* $(\beta_0, \ldots, \beta_g)$.

Every number associated with a characteristic sequence is of form (2) defined in Introduction. Using Theorems 2.2 and 2.3 of this Note we obtain

Theorem 3.1. *Every nonregular Łojasiewicz number is associated with a characteristic sequence. Every rational number strictly associated with a characteristic sequence is a Łojasiewicz number.*

The above result does not give the complete description of the Łojasiewicz numbers. In particular, we cannot replace the assumption 'strictly associated' by 'associated'.

Example 3. The number $\lambda = 169\frac{142}{143}$ is associated with the characteristic sequence $(\beta_0, \beta_1, \beta_2) = (143, 154, 164)$ for $\delta = 308 = 2\bar{\beta}_1$ (uniquely determined by λ) but it is not the Łojasiewicz number.

Using Theorem 3.1 we get the following two results on nonregular Łojasiewicz numbers.

Theorem 3.2. *If a rational number* $\lambda = N + \frac{b}{a}$, $0 < b < a$, $\mathrm{GCD}(a, b) = 1$, $a + b > N$, *is a nonregular Łojasiewicz number then*

(i) a *is a composite* (*i.e. non prime*) *number strictly greater than* 8,
(ii) $a + 6 < \lambda < 2a - 1$.

Theorem 3.3. *For every composite number* $a > 8$ *there exists a nonregular Łojasiewicz number with the smallest denominator equal to* a.

Theorems 3.2 and 3.3 imply that the set of nonregular Łojasiewicz numbers is infinite but the set of such numbers with a fixed denominator is finite. Using Theorem 3.1 one can prove that $\lambda_0 = 15\frac{8}{9}$ is the smallest nonregular Łojasiewicz number.

B. Teissier proposed to use Theorem 3.2 for the construction of non-Jacobian ideals.

Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ be the ring of holomorphic function germs at $0 \in \mathbb{C}^2$. An ideal $I \subset \mathcal{O}$ is called *Jacobian ideal* if there exists a holomorphic function germ f with isolated critical point at 0 such that $I = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\mathcal{O}$. For every ideal $I \subset \mathcal{O}$ of finite codimension we consider the *Łojasiewicz exponent* $\mathcal{L}(I)$ *of* I (see e.g. [3], Remarque 6.2). If $I = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ then $\mathcal{L}(I) = \mathcal{L}_0(f)$.

Using Theorem 3.2 we easily find rational numbers $N + \frac{b}{a}$, $0 < b < a < N$, $a + b > N$, $\mathrm{GCD}(a, b) = 1$, which are not Łojasiewicz numbers. Then the ideals $I = (x^{a+1} - y^a,\ x^{N-b}y^b) \subset \mathcal{O}$ are not Jacobian ideals because $\mathcal{L}(I) = N + \frac{b}{a}$ (see [5], p. 359).

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