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Number Theory

Finiteness of Abelian fundamental groups with restricted ramification

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Abstract

We define a certain quotient of the étale fundamental group of a scheme which classifies étale coverings with bounded ramification along the boundary, and show the finiteness of the abelianization of this group for an arithmetic scheme. ***To cite this article: T. Hiranouchi, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Finitude des groupes fondamentaux abéliens avec ramification bornée. Nous définissons un certain quotient du groupe fondamental étale d'un schéma qui classifie les revêtements étales à ramification bornée le long du bord, et démontrons la finitude de ce groupe rendu abélien pour un schéma arithmétique. ***Pour citer cet article : T. Hiranouchi, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. Introduction

Let X be a connected normal Noetherian scheme, D an effective Weil divisor of X and I the set of irreducible components of D. Put $\mathcal{Q} := \{a, a+ \mid a \in \mathbb{Q}_{\geqslant 1}\}$, where $a+$ is just a formal symbol. For any $\underline{a} = (a_1, \ldots, a_r) \in \mathcal{Q}^I$, we define a fundamental group $\pi_1^{\underline{a}}(X, D)$ which is a quotient of the étale fundamental group $\pi_1(X \setminus D)$. It classifies coverings of X which are étale over $X \setminus D$ and of ramification bounded by $\underline{a}$ along D (see Definition 2.3 below). If the scheme X is regular, then $\pi_1^{\underline{1}}(X, D) = \pi_1(X)$ for $\underline{1} := (1, \ldots, 1)$. For a general X, we have $\pi_1^{\underline{1+}}(X, D) = \pi_1^{\mathrm{tame}}(X, D)$ for $\underline{1+} := (1+, \ldots, 1+)$, where $\pi_1^{\mathrm{tame}}(X, D)$ is the tame fundamental group defined in Exposé XIII

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of [3]. To define $\pi_1^{\underline{a}}(X, D)$, we employ the ramification filtration defined by Abbes and Saito in [1]. Our main theorem is the following.

Theorem 1.1. *Let k be a finite extension of $\mathbb{Q}$ and X a normal scheme of finite type and faithfully flat over the ring of integers $\mathcal{O}$ of k whose geometric generic fiber $X \otimes_{\mathcal{O}} \bar{k}$ is connected. Then, the abelianized fundamental group $\pi_1^{\underline{a}}(X, D)^{\mathrm{ab}}$ is finite for any effective Weil divisor D of X and $\underline{a} \in \mathcal{Q}^I$.*

The above theorem is a generalization of the finiteness result in [6] proved by Katz and Lang for étale fundamental groups, and the recent result in [7] of Schmidt for tame fundamental groups.

Throughout this Note, we assume that all schemes are Noetherian. For any scheme X, we denote by $\mathcal{O}_X$ its structure sheaf. For any field K, we denote by K^{sep} the maximal separable extension of K within a given algebraic closure $\overline{K}$ of K. Finally, we assume that any separable extension of K is contained in K^{sep}.

2. Fundamental groups with restricted ramification

Let K be a complete discrete valuation field, and G_K the absolute Galois group of K. Using techniques of rigid geometry, Abbes and Saito [1] defined a decreasing filtration $(G_K^a)_{a \in \mathbb{Q}_{\geqslant 0}}$ by closed normal subgroups G_K^a of G_K. The filtration coincides with the classical upper numbering ramification filtration shifted by one, if the residue field of K is perfect (see [8], §IV.3 for the classical case). We define G_K^{a+} to be the topological closure of $\bigcup_{b>a} G_K^b$, where b denotes a rational number. In particular, G_K^1 is the inertia subgroup of G_K, and G_K^{1+} is the wild inertia subgroup of G_K.

Definition 2.1. Let L/K be a separable extension. For any $a \in \mathcal{Q}$, we say that the *ramification of L/K is bounded by a* if $G_K^a \subset G_{\tilde{L}}$, where $\tilde{L}$ is the Galois closure of L/K.

This definition is compatible with Definition 6.3 of [1]. Basic properties of the filtration $(G_K^a)_{a \in \mathbb{Q}_{\geqslant 0}}$ imply the following assertions:

Lemma 2.2. *Let L/K and L'/K be separable extensions which have ramification bounded by $a \in \mathcal{Q}$.*

(1) *For any subextension M/K of L/K, the ramification of M/K is bounded by a.*

(2) *The ramification of the composite field LL'/K is bounded by a.*

Let X be a connected normal scheme and Y a normal scheme. We say that a generically étale morphism $Y \to X$ is a *covering* of X if it is finite and every irreducible component of Y dominates X. Let D be an effective Weil divisor of X and $\xi_1, \ldots, \xi_r$ the generic points of the irreducible components of D. Then, the local ring $\mathcal{O}_{X,\xi_i}$ is a discrete valuation ring inducing a discrete valuation v_i on the function field $k(X)$ of X. We denote by $(\mathcal{O}_{X,\xi_i})^\wedge$ its completion with respect to v_i. Let $Y' := Y \times_X \mathrm{Spec}((\mathcal{O}_{X,\xi_i})^\wedge)$. If the covering $Y \to X$ is étale over $X \setminus D$, the total ring of quotients of $\Gamma(Y', \mathcal{O}_{Y'})$ is a finite direct sum of complete discrete valuation fields L_{ij} which are finite separable extensions of the fraction field K_i of $(\mathcal{O}_{X,\xi_i})^\wedge$.

Definition 2.3. Let the notation be as above, and let $\underline{a} = (a_1, \ldots, a_r) \in \mathcal{Q}^I$. The covering $Y \to X$ is said to be of *ramification bounded by $\underline{a}$* along D, if it is étale over $X \setminus D$ and, for each $i = 1, \ldots, r$, the ramification of the extensions L_{ij}/K_i is bounded by a_i for all j.

By the above definition, a covering $Y \to X$ is of ramification bounded by $\underline{1} := (1, \ldots, 1)$ along D if and only if it is étale above points in D of codimension 1 and étale above over $X \setminus D$. Similarly, the ramification of a covering $Y \to X$ is bounded by $\underline{1+} := (1+, \ldots, 1+)$ along D if and only if it is tamely ramified along D in the sense of

Definition 2.2.2 in [4]. Note, however, that this may not be true if we adopt Schmidt's definition of a tame covering (cf. [4], Example 1.3).

In the same way as in Lemma 2.2.5 of [4], we can see that Lemma 2.2 (1) implies the following assertion:

Lemma 2.4. *Let $f : Y \to X$ be a covering, and let $g : Z \to Y$ be a surjective covering. If the ramification of $f \circ g : Z \to X$ is bounded by $\underline{a}$ along D, then so is f.*

Let $\mathbf{Cov}^{\text{ét}}(X)$ be the category of étale coverings of X, and $\mathbf{Cov}^{\underline{a}}(X, D)$ the category of coverings of X which have ramification bounded by $\underline{a}$ along D. The category $\mathbf{Cov}^{\text{ét}}(X)$ is a full subcategory of $\mathbf{Cov}^{\underline{a}}(X, D)$.

As in the proof of Theorem 2.4.2 in [4], Lemmas 2.2(2) and 2.4 imply the existence of fiber products and quotients respectively in the category $\mathbf{Cov}^{\underline{a}}(X, D)$. Choose a point $x \in X$ which is not in D, and take a geometric point $\xi : \operatorname{Spec} \Omega \to x$, where Ω is a separably closed extension of the residue field at x. We define a fiber functor $\mathcal{F}$ by $\mathcal{F}(Y) = \operatorname{Hom}_X(\operatorname{Spec} \Omega, Y)$ for any $Y \in \mathbf{Cov}^{\underline{a}}(X, D)$. Then, we can prove the following theorem:

Theorem 2.5. *The category $\mathbf{Cov}^{\underline{a}}(X, D)$ together with the fiber functor $\mathcal{F}$ is a Galois category.*

Now, we define our fundamental group $\pi_1^{\underline{a}}(X, D; \xi)$ (or simply $\pi_1^{\underline{a}}(X, D)$) to be the fundamental group of this Galois category (cf. Théorème 4.1 in Exposé V of [3]). From Proposition 6.9 in Exposé V of [3], we have the following surjective homomorphisms: $\pi_1(X \setminus D) \to \pi_1^{\underline{a}}(X, D) \to \pi_1(X)$. The category $\mathbf{Cov}^{\underline{1+}}(X, D)$ is the category of tamely ramified coverings of X along D, and we have $\pi_1^{\underline{1+}}(X, D) = \pi_1^{\text{tame}}(X, D)$. If we assume the scheme X is regular, the theorem of Zariski–Nagata on the purity of the branch locus (cf. [3], Exposé X, Théorème 3.1) implies $\mathbf{Cov}^{\underline{1}}(X, D) = \mathbf{Cov}^{\text{ét}}(X)$ and hence $\pi_1^{\underline{1}}(X, D) = \pi_1(X)$.

3. Proof of Theorem 1.1

We basically follow the proof of Schmidt's theorem (cf. [7], Theorem 3.1). For any open subscheme V of $U := X \setminus D$ such that $X \setminus V$ is an effective Weil divisor, there exists a surjective homomorphism $\pi_1^{\underline{b}}(X, X \setminus V) \to \pi_1^{\underline{a}}(X, D)$ for some $\underline{b} \in \mathcal{Q}^J$ and $J \supset I$. Therefore, shrinking U if necessary, we may assume that U is smooth over $\overline{S} := \operatorname{Spec} \mathcal{O}$. Let $S \subset \overline{S}$ be the image of U. There are a surjective homomorphism $\pi_1(U) \to \pi_1^{\underline{a}}(X, D)$ and a natural homomorphism $\pi_1^{\underline{a}}(X, D) \to \pi_1(\overline{S})$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ker}(U/S) & \longrightarrow & \pi_1(U)^{\text{ab}} & \longrightarrow & \pi_1(S)^{\text{ab}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}(X/\overline{S}) & \longrightarrow & \pi_1^{\underline{a}}(X, D)^{\text{ab}} & \longrightarrow & \pi_1(\overline{S})^{\text{ab}}.
\end{array}$$

Here, the groups $\operatorname{Ker}(U/S)$ and $\operatorname{Ker}(X/\overline{S})$ are defined by the exactness of the corresponding rows, and the two right vertical homomorphisms are surjective. By the classical class field theory, the group $\pi_1(\overline{S})^{\text{ab}}$ is finite, and the kernel of the homomorphism $\pi_1(S)^{\text{ab}} \to \pi_1(\overline{S})^{\text{ab}}$ is topologically finitely generated. In addition to this, the group $\operatorname{Ker}(U/S)$ is finite by Theorem 1 of [6]. Since $\pi_1(U)^{\text{ab}}$ and $\pi_1^{\underline{a}}(X, D)^{\text{ab}}$ are topologically finitely generated Abelian groups, it is enough to show that $\operatorname{Ker}(X/\overline{S})$ is torsion. Furthermore, it is known that $\pi_1(U)^{\text{ab}} \to \pi_1(S)^{\text{ab}}$ is surjective by Lemma 2(2) of [6]. By the snake lemma, it is sufficient to show that the cokernel C of $\operatorname{Ker}(U/S) \to \operatorname{Ker}(X/\overline{S})$ is torsion.

Let K be the function field of X, and k' the maximal Abelian extension of k such that the normalization $X_{Kk'}$ of X in Kk' is of ramification bounded by $\underline{a}$ along D. This is equivalent to saying that k' is the compositum of all the finite extensions of k which appear as the fraction fields of the integral closures of S in $\mathcal{O}_Y$ for $Y \to X$

in $\mathbf{Cov}^{\underline{a}}(X, D)$. Note that the normalization of S in k' is ind-étale. Let k'' be the maximal subextension of k'/k such that the normalization $\overline{S}_{k''}$ of $\overline{S}$ in k'' is étale over $\overline{S}$. Then, $\mathrm{Gal}(k''/k) = \pi_1(\overline{S})^{\mathrm{ab}}$ and, by the snake lemma, $C \simeq \mathrm{Gal}(k'/k'')$. To prove the assertion, it is sufficient to show k'/k'' does not contain a $\mathbb{Z}_p$-extension of k'' for any prime number p. Since k''/k is a finite extension and k'/k is Abelian, it is enough to show that k'/k does not contain a $\mathbb{Z}_p$-extension. So, we assume that k'/k contains a $\mathbb{Z}_p$-extension k_∞/k. A $\mathbb{Z}_p$-extension of k is unramified outside p and at least ramified at one prime $\mathfrak{p}$ dividing p (cf. [5], §6, Lemma 4). Since the normalization of S in k' is ind-étale, $\mathfrak{p} \in \overline{S} \setminus S$. From the assumption, the prime $\mathfrak{p}$ is in the image of $X \to \overline{S}$. By the definition of k', the normalization of X in Kk_∞ has ramification bounded by $\underline{a}$ along D. This carries over to the local situation, which contradicts the following lemma:

Lemma 3.1. *Let R be a complete discrete valuation ring with fraction field k of characteristic* 0 *and perfect residue field of characteristic $p > 0$. Let X be a normal faithfully flat scheme of finite type over R whose geometric generic fiber is connected, and D a Weil divisor of X containing an irreducible component of the closed fiber $X_{\mathfrak{p}}$ of X. Then, for a ramified $\mathbb{Z}_p$-extension k_∞ of k, the ramification of $X \otimes_{\mathcal{O}_k} \mathcal{O}_{k_\infty} \to X$ is not bounded along D.*

For any point $\mathfrak{P} \in D \cap X_{\mathfrak{p}}$ of codimension 1, let K be the completion of the function field $k(X)$ at $\mathfrak{P}$. We assume the ramification of Kk_∞/K is bounded by some $a \in \mathbb{Q}$. By Theorem 1.9 of [2], there exists a finite extension $\tilde{k}/k$ such that the extension $K\tilde{k}$ over $\tilde{k}$ is *weakly unramified*, i.e., a uniformizing element of $\tilde{k}$ is a uniformizing element of $K\tilde{k}$. Lemma 6.5 of [1] implies the ramification of the composite field $K\tilde{k}k_\infty$ over $K\tilde{k}$ is bounded by ae, where e is the ramification index of $K\tilde{k}/K$. Changing the base field from k to $\tilde{k}$, we shall consider the problem over $\tilde{k}$; thus we write k, k_∞, K, etc., instead of $\tilde{k}, \tilde{k}k_\infty, \tilde{k}K$, etc. Hence, the extension K/k is regular and weakly unramified. Replacing k by the maximal unramified subextension of k_∞/k, we may suppose k_∞/k is totally ramified. Let k_n be the unique subextension of k_∞/k such that the extension degree is p^n over k. Since the extension K/k is regular, we have $\mathrm{Gal}(Kk_n/K) \simeq \mathrm{Gal}(k_n/k)$, and $K \otimes_k k_n \simeq Kk_n$. Then, an Eisenstein polynomial $f \in \mathcal{O}_k[T]$ for the extension k_n/k remains to be Eisenstein over K, and we have $\mathcal{O}_{Kk_n} = \mathcal{O}_K[T]/(f)$. In this case, the differents $\mathfrak{D}_{k_n/k}$ of k_n/k and $\mathfrak{D}_{Kk_n/K}$ of Kk_n/K are both generated by $f'(\pi_n)$ for some uniformizing element π_n of k_n, and we have $v_k(\mathfrak{D}_{k_n/k}) = v_K(\mathfrak{D}_{Kk_n/K})$, where v_k, v_K are the normalized valuations of k, K, respectively. Lemma 6.6 of [1] says that, if the ramification of Kk_n/K is bounded by $a \in \mathbb{Q}$, then $a > v_K(\mathfrak{D}_{Kk_n/K})$. However, $v_k(\mathfrak{D}_{k_n/k})$ tends to infinity as $n \to \infty$ (cf. [9], §3, Proposition 5).

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