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## Partial Differential Equations

# Liouville-type results for solutions of $-\Delta u=|u|^{p-1} u$ on unbounded domains of $\mathbb{R}^{N}$ 

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#### Abstract

In this Note we study solutions, possibly unbounded and sign-changing, of the equation $-\Delta u=|u|^{p-1} u$ on unbounded domains of $\mathbb{R}^{N}$ with $N \geqslant 2$ and $p>1$. We prove some Liouville-type results and a classification theorem for $C^{2}$ solutions belonging to one of the following classes: stable solutions, finite Morse index solutions and solutions which are stable outside a compact set. We also extend, to smooth coercive epigraphs, the well-known results of Gidas and Spruck concerning non-negative solutions of the considered equation. To cite this article: A. Farina, C. R. Acad. Sci. Paris, Ser. I 341 (2005). © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.


## Résumé

Résultats de type Liouville pour des solutions de $-\Delta u=|u|^{p-1} u$ dans des domaines non-bornés de $\mathbb{R}^{N}$. Cette Note porte sur l'étude des solutions, éventuellement non-bornées et de signe quelconque, de l'équation $-\Delta u=|u|^{p-1} u$ dans des domaines non-bornés de $\mathbb{R}^{N}$ avec $N \geqslant 2$ et $p>1$. Nous démontrons des résultats de type Liouville ainsi que des théorèmes de classification pour les solutions régulières appartenant à une des classes suivantes : solutions stables, solutions d'indice de Morse fini et solutions stables à l'extérieur d'un compact. Nous étendons aussi, au cas d'un épigraphe coercif régulier, les célèbres résultats de Gidas et Spruck concernant les solutions positives de l'équation considérée. Pour citer cet article :A. Farina, C. R. Acad. Sci. Paris, Ser. I 341 (2005).
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## 1. Introduction

This Note is devoted to the study of solutions (possibly unbounded and sign-changing) of the semilinear partial differential equation :

$$
\begin{equation*}
-\Delta u=|u|^{p-1} u \quad \text { in } \Omega, \tag{1}
\end{equation*}
$$

where $p>1, \Omega$ is an unbounded domain of $\mathbb{R}^{N}$ with $N \geqslant 2$.

[^0]There is an extensive literature on this type of equations on bounded domains. We refer to [1,3,12,13] and the references therein. Our purpose is to consider this problem in the whole space or on some unbounded domains like half spaces and coercive epigraphs. More specifically, we prove some Liouville-type results as well as a classification theorem for solutions $u \in C^{2}(\Omega)$ of (1) belonging to one of the following classes (see definition below): stable solutions, finite Morse index solutions, solutions which are stable outside a compact set of $\Omega$ and non-negative solutions.

Eq. (1) naturally arises both in physics and in geometry. Furthermore, Liouville-type results play a crucial role to obtain a priori $L^{\infty}$-bounds for solutions of semilinear boundary value problems in bounded domains (see [9] for the case of positive solutions and [2] for solutions having finite Morse index). The proofs of our results will appear in the forthcoming work [7]. This paper also contains some further results concerning qualitative properties of solutions of (1) as well as some extensions to the non-autonomous case.

## 2. Main results

In order to state our results we need to recall the following:
Definition 2.1. We say that a solution $u$ of (1) belonging to $C^{2}(\Omega)$

- is stable if:

$$
\forall \psi \in C_{c}^{1}(\Omega) \quad Q_{u}(\psi):=\int_{\Omega}|\nabla \psi|^{2}-p|u|^{p-1} \psi^{2} \geqslant 0,
$$

- has Morse index equal to $K \geqslant 1$ if $K$ is the maximal dimension of a subspace $X_{K}$ of $C_{c}^{1}(\Omega)$ such that $Q_{u}(\psi)<0$ for any $\psi \in X_{K} \backslash\{0\}$,
- is stable outside a compact set $\mathcal{K}$ of $\Omega$ if $Q_{u}(\psi) \geqslant 0$ for any $\psi \in C_{c}^{1}(\Omega \backslash \mathcal{K})$.

Remark 1. Any finite Morse index solution $u$ is stable outside a compact set of $\Omega$.
Our main results are:
Theorem 2.2. Let $u \in C^{2}\left(\mathbb{R}^{N}\right)$ be a stable solution of (1) with:

$$
\begin{cases}1<p<+\infty & \text { if } N \leqslant 4 \\ 1<p \leqslant \frac{N}{N-4} & \text { if } N>4\end{cases}
$$

Then $u \equiv 0$.
Remark 2. Note that the exponent $\frac{N}{N-4}(=+\infty$ if $N \leqslant 4)$ is larger than the classical critical exponent $\frac{N+2}{N-2}\left(=2^{\star}-1\right)$ when $N \geqslant 3$.

Theorem 2.3. Let $u \in C^{2}\left(\mathbb{R}^{N}\right)$ be a solution of (1) which is stable outside a compact set of $\mathbb{R}^{N}$. Suppose

$$
\begin{cases}1<p<+\infty & \text { if } N \leqslant 2, \\ 1<p<\frac{N+2}{N-2} & \text { if } N>2,\end{cases}
$$

then $u \equiv 0$. On the other hand, if $N>2$ and $p=\frac{N+2}{N-2}$ then

$$
\int_{\mathbb{R}^{N}}|\nabla u|^{2}=\int_{\mathbb{R}^{N}}|u|^{2 N /(N-2)}<+\infty .
$$

Remark 3. (i) Theorem 2.3 is sharp. Indeed, for any $N \geqslant 3$ the function $u(x)=\left(\frac{\sqrt{N(N-2)}}{1+|x|^{2}}\right)^{(N-2) / 2}$ solves Eq. (1), with the classical critical exponent $p=\frac{N+2}{N-2}$, and is stable outside a large ball centered at the origin using Hardy's inequality.
(ii) Theorem 2.3 improves upon the Liouville-type result proved in [2] where solutions are assumed to be both bounded and with finite Morse index (with $1<p<\frac{N+2}{N-2}$ if $N>2, p<+\infty$ if $N=2$ ).

The proofs of the above theorems are based on the following crucial proposition.
Proposition 2.4. Let $u \in C^{2}(\Omega)$ be a stable solution of (1) with $p>1$. Then, for any $m \geqslant \max \left\{\frac{p+1}{p-1}, 2\right\}$ there exists $a$ constant $C_{p, m}>0$, depending only on $p$ and $m$, such that:

$$
\left\{\psi \in C_{c}^{2}(\Omega),|\psi| \leqslant 1 \text { in } \Omega\right\} \Longrightarrow \int_{\Omega}\left(|\nabla u|^{2}+|u|^{p+1}\right) \psi^{2 m} \leqslant C_{p, m} \int_{\Omega}\left(|\nabla \psi|^{2}+|\psi||\Delta \psi|\right)^{(p+1) /(p-1)}
$$

The next result concerns the complete classification of entire radial solutions of (1) which are stable outside a compact set of $\mathbb{R}^{N}$. The proof is based on a combination of Hardy's inequality, Theorem 2.3 and some well-know results about positive smooth entire radial solutions of Eq. (1) (see for instance $[8,10,11,14]$ ).

Theorem 2.5. Let $u \in C^{2}\left(\mathbb{R}^{N}\right)$ be a radial solution $(u \not \equiv 0)$ of $(1)$ which is stable outside a compact set of $\mathbb{R}^{N}$. Then $u$ does not change sign (i.e. either $u>0$ or $u<0$ everywhere).

Moreover only two cases occur:
(a) $N \geqslant 3, p=\frac{N+2}{N-2}, u(x)=\epsilon\left(\frac{\lambda \sqrt{N(N-2)}}{\lambda^{2}+|x|^{2}}\right)^{(N-2) / 2}$ with $\lambda>0, \epsilon \in\{-1,1\}$,
(b) $N \geqslant 11, p \geqslant p_{c}:=\frac{(N-2)^{2}-4 N+8 \sqrt{N-1}}{(N-2)(N-10)}, u$ is stable and of the form:

$$
\forall x \in \mathbb{R}^{N} \quad u_{\alpha}(x)=\epsilon \alpha^{2 /(p-1)} v(\alpha|x|)
$$

with $\alpha>0, \epsilon \in\{-1,1\}$. The profile $v$ satisfies: $v(0)=1, v>0, v^{\prime}<0$ in $\mathbb{R}_{*}^{+}$.
Remark 4. Note that $p_{c}:=\frac{(N-2)^{2}-4 N+8 \sqrt{N-1}}{(N-2)(N-10)}>\frac{N}{N-4}$.
In this regard we mention the recent paper [4], where the authors establish that any smooth bounded stable radial entire solution of $-\Delta u=f(u)$ is constant if $N \leqslant 10$ and $f$ is a $C^{1}$ function satisfying a generic nondegeneracy condition.

The next theorem deals with non-negative solutions of (1). It extends the celebrated results of Gidas and Spruck $[9,10]$ to the case where the unbounded domain $\Omega$ is a coercive epigraph. Recall that a domain $\Omega$ is a smooth coercive epigraph if $\Omega:=\left\{\left(x^{\prime}, x_{N}\right) \in \mathbb{R}^{N}: \varphi\left(x^{\prime}\right)<x_{N}\right\}$ where $\varphi$ belongs to $C_{\text {loc }}^{2, \alpha}\left(\mathbb{R}^{N-1}, \mathbb{R}\right)(0<\alpha<1)$ and satisfies $\lim _{\left|x^{\prime}\right| \rightarrow+\infty} \varphi\left(x^{\prime}\right)=+\infty$.

Theorem 2.6. Let $\Omega$ be a smooth coercive epigraph. Let $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ be a solution of:

$$
\begin{cases}-\Delta u=|u|^{p-1} u & \text { in } \Omega  \tag{2}\\ u \geqslant 0 & \text { on } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

with

$$
\begin{cases}1<p<+\infty & \text { if } N \leqslant 2 \\ 1<p \leqslant \frac{N+2}{N-2} & \text { if } N>2\end{cases}
$$

Then $u \equiv 0$.

The proof of Theorem 2.6 is based on the observation that any solution $u$ of (2) is automatically stable; then a variant of Proposition 2.4 above implies the desired conclusion. To prove the stability of $u$ we proceed as follows: by the strong minimum principle either $u \equiv 0$, and then $u$ is stable, or $u>0$ in $\Omega$ and then $\frac{\partial u}{\partial x_{N}}>0$ in $\Omega$ (by Proposition II. 1 of [6]). The latter property then easily implies the stability of $u$.

Using similar arguments together with Theorem 2.2 we can prove:

Theorem 2.7. Let $\Omega$ be either a half-space or a smooth coercive epigraph. Let $u \in C^{2}(\Omega) \cap C^{0}(\bar{\Omega})$ be a bounded non-negative solution of:

$$
\begin{cases}-\Delta u=u^{p} & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega,\end{cases}
$$

with

$$
\begin{cases}1<p<+\infty & \text { if } N \leqslant 5, \\ 1<p \leqslant \frac{N-1}{N-5} & \text { if } N>5 .\end{cases}
$$

Then $u \equiv 0$.
Remark 5. (i) Note that, when $\Omega$ is a half-space, Theorem 2.7 improves upon a result proved in [5] where the exponent $p$ was assumed to satisfy $1<p<\frac{N+1}{N-3}$ if $N>3(p<+\infty$ if $N \leqslant 3)$.
(ii) Theorem 2.7 also holds for more general unbounded domains (see [7]).

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## Further reading

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