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An asymptotically stable discretization for the Euler–Poisson system in the quasi-neutral limit

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Abstract

We are interested in the modeling of a plasma in the quasi-neutral limit using the Euler–Poisson system. When this system is discretized with a standard numerical scheme, it is subject to a severe numerical constraint related to the quasi-neutrality of the plasma. We propose an asymptotically stable discretization of this system in the quasi-neutral limit. We present numerical simulations for two different one-dimensional test cases that confirm the expected stability of the scheme in the quasi-neutral limit. ***To cite this article: P. Crispel et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Une discrétisation asymptotiquement stable dans la limite quasi-neutre, pour le système d'Euler–Poisson. On s'intéresse à la modélisation d'un plasma dans la limite quasi-neutre à l'aide du système d'Euler–Poisson. Lorsque ce système est discrétisé par une méthode numérique standard, il est sujet à une contrainte numérique sévère liée à la quasi-neutralité dans le plasma. Nous proposons une discrétisation asymptotiquement stable dans la limite quasi-neutre. Nous présentons des simulations numériques de deux cas tests monodimensionnels qui confirment la stabilité attendue du schéma dans la limite quasineutre. ***Pour citer cet article : P. Crispel et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

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Version française abrégée

Nous nous intéressons à la simulation numérique d'un plasma dans la limite quasi-neutre à l'aide de modèles fluides pour chaque espèce (ions et électrons). Pour cela, nous introduisons les densités et vitesses ioniques et électroniques, n_i, u_i and n_e, u_e, ainsi que le potentiel électrostatique ϕ. Ces quantités macroscopiques satisfont les équations d'Euler isentropiques pour chaque espèce couplées à l'équation de Poisson, qui en variables adimensionnées sont données par (4)–(6).

Ce modèle est bien connu pour présenter des échelles spatiales et temporelles très petites : la longueur de Debye λ_D et la fréquence plasma électronique ω_p données par [1] :

$$\lambda_D = \left(\frac{\varepsilon_0 T}{q^2 n}\right)^{1/2}, \qquad \omega_p = \left(\frac{q^2 n}{\varepsilon_0 m_e}\right)^{1/2} \tag{1}$$

où T est la température, n, la densité dans le plasma, $q > 0$, la charge élémentaire, ε_0, la permittivité du vide et m_e, la masse électronique. Il est admis dans la littérature physiques que toute discrétisation numérique du système d'Euler–Poisson doit résoudre ces deux échelles, et ce, y compris si le terme source dû au champ électrique est traité de manière implicite.

Or dans beaucoup d'applications où les densités de plasma sont très élevées, λ_D est très petit et ω_p très grand, ce qui rend les simulations numériques trop coûteuses. Pour résoudre ce problème, la limite quasi-neutre, $n_i = n_e$, est fréquemment utilisée. Elle consiste à faire $\lambda = \lambda_D/L \to 0$ dans le système Euler–Poisson, où L est l'échelle de longueur typique du problème. Cependant, il existe des situations (voir par exemple [2,3]) où λ ne tend pas vers 0 uniformément dans le domaine. Il convient dans ces cas là de développer une méthode numérique qui s'affranchit des contraintes liées à λ_D et ω_p. C'est l'objet de cet article. Plus précisément, nous proposons une méthode numérique asymptotiquement stable pour le système Euler–Poisson à la limite quasi-neutre $\lambda \to 0$ *qui a le même coût qu'une méthode classique*.

Cette méthode est basée sur la remarque que l'on peut reformuler l'équation de Poisson à l'aide des équations d'Euler comme :

$$-\nabla \cdot \left(\lambda^2 \nabla \partial_{tt}^2 \phi + \left(n_i + \frac{n_e}{\varepsilon}\right)\nabla\phi\right) = \nabla^2 : \big(f_i(n_i, u_i) - f_e(n_e, u_e)\big), \tag{2}$$

où les flux f_i, f_e sont explicités plus loin. Pour tout $\lambda > 0$, cette équation est une forme équivalente de l'équation de Poisson sous l'hypothèse que les équations d'Euler (4), (5) soient vérifiées et qu'à $t = 0$, on ait simultanément

$$(-\lambda^2 \Delta\phi - \rho)_{t=0} = 0 \quad \text{et} \quad \left(\frac{\mathrm{d}}{\mathrm{d}t}(-\lambda^2 \Delta\phi - \rho)\right)_{t=0} = 0. \tag{3}$$

Dans la limite quasineutre $\lambda \to 0$ on obtient encore un problème elliptique (donc bien-posé).

Le schéma numérique asymptotiquement stable que nous proposons repose sur la remarque que celui-ci doit conduire à une discrétisation implicite en temps de (2). Pour cela, nous discrétisons le système d'Euler–Poisson en implicitant les flux dans les équations de conservation de masse (voir (8), (9)). Nous reformulons l'équation discrète associée au potentiel en utilisant les mêmes manipulations que dans le cas continu (voir (12)). Là encore, celle-ci est équivalente pour tout $\lambda > 0$ à la discrétisation classique de l'équation de Poisson et reste bien posée dans la limite quasineutre $\lambda \to 0$. De plus, elle permet de calculer successivement le potentiel, puis les vitesses, et enfin les densités, par une succession de calculs explicites, ce qui rend la méthode d'un coût égal à une méthode explicite classique pour le modèle d'Euler–Poisson.

Nous comparons le schéma ainsi obtenu à des discrétisation classiques du système d'Euler–Poisson au travers de deux cas tests. Le premier concerne une perturbation d'un plasma uniforme et stationnaire. Le second cas test est le suivi de l'expansion d'un plasma quasi-neutre entre deux électrodes. Les résultats des Figs. 1 et 2 montrent le bon comportement asymptotique de notre schéma contrairement aux schémas classiques.

1. Introduction

We are interested in the numerical simulation of a plasma in the quasi-neutral limit by means of fluid models. The plasma is described by the ion and electron densities and velocities, n_i, u_i and n_e, u_e, and by the electric potential ϕ. These macroscopic quantities satisfy the isentropic Euler equations coupled to Poisson equation. The isentropic assumption can be removed and replaced by energy equations without any additional difficulty.

It is well known [1] that this model presents fast time and length scales related to the Debye length λ_D and to the plasma frequency ω_p given by (1), where T is the temperature, n, the density, $q > 0$, the elementary charge, ε_0, the vacuum permittivity and m_e the electron mass. It is common wisdom in plasma physics that numerical discretizations of this model must resolve these scales, even if the source terms due to the electric field are treated implicitly.

In many applications, the densities are very large which result in a small Debye length and a large plasma frequency. To bypass this limitation, the quasi-neutral model ($n_i = n_e$) is commonly used. It is obtained from the Euler–Poisson system by passing to the limit $\lambda := \lambda_D/L \to 0$ where L is the typical length of the problem. However, in some situations, the quasineutral limit is only relevant in a subdomain of the simulation domain [2,3]. For these problems, we need a numerical method for the Euler–Poisson system which is not constrained to resolve the Debye length and plasma frequency. The goal of this Note is to propose such a method.

More precisely, we propose a numerical method for the Euler–Poisson problem which is asymptotically stable in the quasi-neutral limit $\lambda \to 0$, *and which has the same numerical cost as a standard method*. This method is based on a reformulation of the Poisson equation using the Euler equations, as shown in the next section.

2. Reformulation of the Euler–Poisson system

The Euler–Poisson system, in scaled variables, is given by

$$\partial_t n_i + \nabla\cdot(n_i u_i) = 0, \qquad \partial_t n_e + \nabla\cdot(n_e u_e) = 0, \tag{4}$$

$$\partial_t(n_i u_i) + \nabla(n_i u_i\otimes u_i) + \nabla p_i(n_i) = -n_i\nabla\phi, \qquad \varepsilon\big(\partial_t(n_e u_e) + \nabla(n_e u_e\otimes u_e)\big) + \nabla p_e(n_e) = n_e\nabla\phi, \tag{5}$$

$$-\lambda^2\Delta\phi = n_i - n_e = \rho, \tag{6}$$

where $p_{e,i}(n) = C_{e,i} n^{\gamma_{e,i}}$ are the given pressure laws ($C_{e,i} > 0$, $\gamma_{e,i} > 1$), $\varepsilon = m_e/m_i$ is the mass ratio and $\lambda = \lambda_D/L$ is the rescaled Debye length. For future usage, we introduce $q_{i,e} = n_{i,e}u_{i,e}$.

Let us first remark that the quasi-neutral limit is a dispersive limit. Indeed, let us suppose that $n_i = 1$ and $u_i = 0$. We look for solutions of the linearized system about the steady state $n_e = 1$, $u_e = 0$, $\nabla\phi = 0$ in a WKB form: $n_e(x,t) = A(x,t)\,\mathrm{e}^{\mathrm{i}S(x,t)/\lambda}$. When $\lambda \ll 1$, we obtain approximate equations for A and S in the form of a pair of eiconal and transport equations, namely:

$$\varepsilon|\partial_t S|^2 - T|\nabla S|^2 - 1 = 0, \quad \partial_t W + \nabla(u_G W) = 0, \quad u_G = (T/\varepsilon)\nabla S/\partial_t S,$$

where $W = A^2$ and $T = (\mathrm{d}p_e/\mathrm{d}n)(1)$. Therefore, in the quasi-neutral limit $\lambda \to 0$ there subsist solutions of amplitude $\mathrm{O}(1)$ which are oscillating with frequency $1/\lambda$. Explicit discretizations might undergo instabilities if the mesh does not resolve these solutions (even if their amplitude is small). The goal of this Note is to propose a scheme without this limitation.

For that purpose, we reformulate the Poisson equation by taking the time derivative of the difference between the mass conservation equations (4) as well as the space divergence of the difference between the momentum conservation equations (5) and subtracting the resulting equations. We obtain:

$$\partial_{tt}^2\rho - \nabla^2 : \big(f_i(n_i,u_i) - f_e(n_e,u_e)\big) = \nabla\cdot\left(\left(n_i + \frac{n_e}{\varepsilon}\right)\nabla\phi\right), \tag{7}$$

where $f_i(n_i,u_i)=n_iu_i\otimes u_i+p_i(n_i)\mathrm{Id}$ and $f_e(n_e,u_e)=n_eu_e\otimes u_e+\frac{1}{\varepsilon}p_e(n_e)\mathrm{Id}$ and : denotes the contracted product of two tensors. Then, eliminating ρ by using the Poisson equation (6), we obtain (2).

We remark that for all $\lambda>0$, Eq. (2) is equivalent to the Poisson equation if (4), (5) are satisfied and if the problem is initiated such that at $t=0$ we have (3). In the quasi-neutral limit $\lambda=0$ the resulting equation for the potential is still a well-posed elliptic equation. Our asymptotically stable numerical scheme relies on the remark that it must lead to a time-implicit discretization of (2). For this purpose, we use an implicit evaluation of the discrete mass fluxes in (4) and we reformulate the discrete Poisson equation using the same manipulations as those which lead to (2), as summarized in the next section.

3. An asymptotically stable discretization for the Euler–Poisson system

The domain $[0,1]$ is discretized by a uniform regular grid of size Δx. We denote by $N=1/\Delta x$ the number of cells.

To be able to perform the same algebra as in the continuous case, we need a scheme where the discrete mass flux is a simple function of the momentum variables. This rules out all schemes based on Riemann solvers and therefore, we resorted to a semi-implicit modified Lax–Friedrichs scheme, in spite of its large numerical diffusion. We set $(U_{i,e})_k^n=((n_{i,e})_k^n,(q_{i,e})_k^n)$ and $(U_{i,e})_k^{n+1/2}=((n_{i,e})_k^n,(q_{i,e})_k^{n+1})$ where k and n are the space and time discretization indices. Then the discretized equations associated with (4) and (5) are given by

$$\frac{(n_i)_k^{n+1}-(n_i)_k^n}{\Delta t}+\frac{1}{\Delta x}\big[Q_i^n\big((U_i)_k^{n+1/2},(U_i)_{k+1}^{n+1/2}\big)-Q_i^n\big((U_i)_{k-1}^{n+1/2},(U_i)_k^{n+1/2}\big)\big]=0, \tag{8}$$

$$\frac{(q_i)_k^{n+1}-(q_i)_k^n}{\Delta t}+\frac{1}{\Delta x}\big[F_i^n\big((U_i)_k^n,(U_i)_{k+1}^n\big)-F_i^n\big((U_i)_{k-1}^n,(U_i)_k^n\big)\big]=-(n_i)_k^n\frac{\phi_{k+1}^{n+1}-\phi_{k-1}^{n+1}}{2\Delta x}, \tag{9}$$

for the ions and similarly (mutatis mutandis) for the electrons, where the numerical fluxes are the following for $l=i$ or e:

$$Q_l^n(U_g,U_d)=\frac{q_g+q_d}{2}+\Lambda_l^n(n_g-n_d)\quad\text{and}\quad F_l^n(U_g,U_d)=\frac{f_l(n_g,u_g)+f_l(n_d,u_d)}{2}+\Lambda_l^n(q_g-q_d). \tag{10}$$

The upwind constants Λ_i^n and Λ_e^n, are chosen in order to ensure the consistency of the scheme (see [4]): for $l=i$ or e, we set $\Lambda_l^n=\frac{1}{2}\max\{|(u_l)_k^n\pm\sqrt{p_l'((n_l)_k^n)}\,|;\ k=1,\dots,N\}$.

It remains to determine the discrete potential ϕ_k^{n+1}. The classically used discretization of the Poisson equation is given by:

$$-\lambda^2\Delta_{ap}(\phi_k^n):=-\lambda^2\frac{\phi_{k+1}^n-2\phi_k^n+\phi_{k-1}^n}{\Delta x^2}=\rho_k^n. \tag{11}$$

The implicitness of the flux term in the mass equation (8) allows us to rewrite it, following the same procedure as in the continuous case, according to:

$$\begin{aligned}&-\lambda^2\frac{\Delta_{ap}(\phi_k^{n+1})-2\Delta_{ap}(\phi_k^n)+\Delta_{ap}(\phi_k^{n-1})}{\Delta t^2}\\&\quad-\frac{(n_i+n_e/\varepsilon)_{k+1}^n(\phi_{k+2}^{n+1}-\phi_k^{n+1})/(2\Delta x)-(n_i+n_e/\varepsilon)_{k-1}^n(\phi_k^{n+1}-\phi_{k-2}^{n+1})/(2\Delta x)}{2\Delta x}\\&\quad=\frac{(f_i-f_e)_{k+2}^n-2(f_i-f_e)_k^n+(f_i-f_e)_{k-2}^n}{4\Delta x^2}-(D_i)_k^n+(D_e)_k^n,\end{aligned} \tag{12}$$

where $(f_i - f_e)_k^n = f_i((n_i)_k^n, (u_i)_k^n) - f_e((n_e)_k^n, (u_e)_k^n)$ and where for $l = i$ or e, $(D_l)_k^n$ is an upwind term given by $(D_l)_k^n = \Delta x[\Lambda_l^n \Delta_{ap}((n_l)_k^n) - \Lambda_l^{n-1} \Delta_{ap}((n_l)_k^{n-1})]/\Delta t + \Lambda_l^n \Delta x[\Delta_{ap}((q_l)_{k+1}^n) - \Delta_{ap}((q_l)_{k-1}^n)]/(2\Delta x)$. We remark that Eq. (12) gives an implicit discretization of (2).

The overall scheme (8), (9), (12) can actually be solved *explicitly*, i.e. with *no additional cost* compared with a method with an explicit treatment of the mass flux. Indeed, assuming all data are known at time step n, (12) provides ϕ^{n+1} by inverting an elliptic operator (this being just as difficult as solving the Poisson equation, which has to be done anyway with an explicit method). Then, the momenta equation (9) can be updated explicitly for finding $q_{i,e}^{n+1}$. Finally, the mass equation (8) can be updated explicitly for finding $n_{i,e}^{n+1}$.

In the next section, we experimentally show that the method has the expected behaviour. The mathematical analysis is under current investigation.

4. Numerical results

We compare the numerical simulations of the Euler–Poisson system for different discretizations. First, we consider a classical discretization using the modified Lax–Friedrichs scheme with explicit mass flux terms and implicit treatment of the electric field source term, referred to as the "explicit Lax–Friedrichs scheme". Then, still using an explicit treatment of the mass flux term and an implicit treatment of the electric field source term, we consider a Godunov scheme, later referred to as "explicit Godunov scheme" (in both cases, the word 'explicit' refers to the treatment of the mass flux term and in both cases, the electric field source terms are discretized implicitly). The comparison with the Godunov scheme is made in order to measure the influence of the numerical viscosity of the Lax–Friedrichs scheme. We compare these two classical schemes to our 'asymptotically stable scheme' (8), (9), (12).

We consider two test cases for which $\gamma_i = \gamma_e = 5/3$, $C_i = C_e = 1$, $\varepsilon = 10^{-4}$ and $\lambda = 10^{-4}$. Note that the dimensionless plasma frequency is $\omega_p = 1/(\sqrt{\varepsilon}\lambda) = 10^6$.

The first test case is a perturbation about a uniform quasineutral stationary solution to the Euler–Poisson system. The initial condition is $n_i = n_e = 1$, $q_e = 1 + 10^{-2}\cos(2\pi x)$, $q_i = 10^{-2}\cos(2\pi x)$, $\phi = 0$ for all $x \in [0, 1]$ and the boundary conditions are periodic. In this case, the analytical solution of the linearized system is known [2].

The second test case models the expansion of a plasma between two electrodes. The initial condition is $n_e = n_i = 0$, $q_e = q_i = 0$ and the boundary conditions at $x = 0$ are $n_i = n_e = 1$, $q_i = q_e = 1$, $\phi = 0$ and at $x = 1$, $\phi = 100$. Solutions for this case have been provided in [2].

Fig. 1 shows the electron density for the first test case at the rescaled time $t = 10^{-2}$. On the left figure, the explicit Lax–Friedrichs and Godunov schemes as well as the asymptotically stable scheme are used while satisfying the

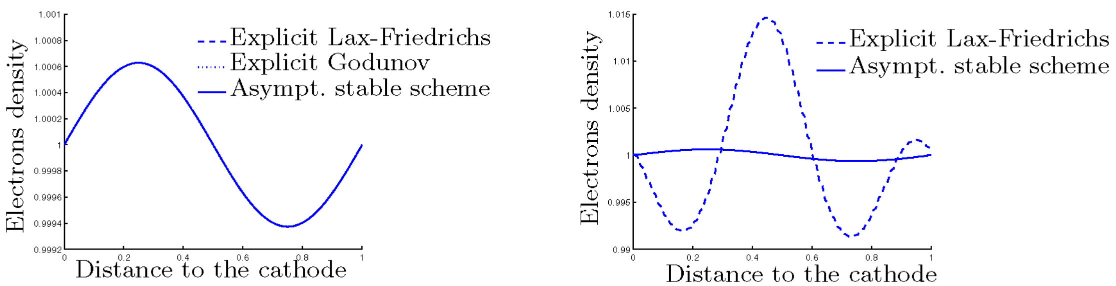

Fig. 1. Perturbation of a uniform plasma: electron density, at time $t = 10^{-2}$. Left: explicit Lax–Friedrichs and Godunov schemes and asymptotically stable scheme with $\Delta x = 10^{-4}$ and $w_p \Delta t \leqslant 1$ (Debye length and plasma frequency are *fully resolved*). Right: explicit Lax–Friedrichs scheme (explicit Godunov scheme fails) and asymptotically stable scheme with $\Delta x = 10^{-2}$ and $w_p \Delta t > 1$ (Debye length and plasma frequency are *not resolved*).

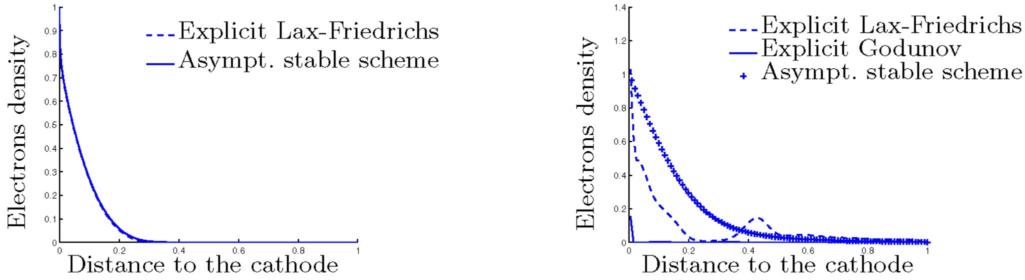

Fig. 2. Plasma expansion: electron density, at time $t = 5 \times 10^{-1}$. Left: explicit Lax–Friedrichs scheme and asymptotically stable scheme with $\Delta x = 10^{-4}$ and $w_p \Delta t \leqslant 1$ (Debye length and plasma frequency are *fully resolved*). Right: explicit Lax–Friedrichs and Godunov schemes and asymptotically stable scheme with $\Delta x = 10^{-2}$ and $w_p \Delta t > 1$ (Debye length and plasma frequency are *not resolved*).

constraints $\Delta x \leqslant \lambda$ and $\omega_p \Delta t \leqslant 1$. Furthermore, we can see on the right figure, the 'asymptotically stable scheme' when the constraints $\Delta x \leqslant \lambda$ and $\omega_p \Delta t \leqslant 1$ are *not* satisfied. We see that the asymptotically stable scheme behaves well, even when it does not resolve the smallest scales of the problem. By contrast, the explicit Lax–Friedrichs and Godunov schemes become unstable when these scales are left unresolved, as it is shown on the right-side figures.

Fig. 2 shows the electron density for the plasma expansion test case at the rescaled time $t = 5 \times 10^{-1}$, with the different schemes. On the left figure, the constraints $\Delta x \leqslant \lambda$, $\omega_p \Delta t \leqslant 1$ are satisfied and on the right figure, they are not. Again, we clearly see the expected good behavior of the asymptotically stable scheme in undersolved conditions while the explicit Lax–Friedrichs and Godunov schemes collapse. Some numerical diffusion is noticeable (and is magnified by the use of the large time steps), which is to be expected from the use of a Lax–Friedrichs scheme. More investigations are necessary to adapt this methodology to Godunov schemes. The computing time for the three schemes are exactly the same.

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