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Complex Analysis

On boundaries of Levi-flat hypersurfaces in $\mathbb{C}^n$

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Abstract

Let S be a smooth 2-codimensional real compact submanifold of $\mathbb{C}^n$, $n > 2$. We address the problem of finding a compact hypersurface M, with boundary S, such that $M \setminus S$ is Levi-flat. We prove the following theorem. Assume that (i) S is nonminimal at every CR point, (ii) every complex point of S is flat and elliptic and there exists at least one such point, (iii) S does not contain complex submanifolds of dimension $n-2$. Then there exists a Levi-flat $(2n-1)$-subvariety $\widetilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with negligible singularities and boundary $\widetilde{S}$ (in the sense of currents) such that the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a CR diffeomorphism between S and $\widetilde{S}$. ***To cite this article: P. Dolbeault et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Bords d'hypersurfaces Levi-plates dans $\mathbb{C}^n$. Soit S une sous-variété réelle, lisse. compacte, de codimension 2 de $\mathbb{C}^n$, $n > 2$. On considère le problème de l'existence d'une hypersurface compacte M, de bord S, telle que $M \setminus S$ soit Levi-plate. On démontre le théorème suivant : supposons que (i) S est non minimale en tout point CR, (ii) tout point complexe de S est plat et elliptique et il en existe un au moins, (iii) S ne contient aucune sous-variété complexe de dimension $n-2$. Alors il existe une sous-variété $\widetilde{M} \subset \mathbb{C} \times \mathbb{C}^n$, à singularités négligeables, avec bord $\widetilde{S}$ (au sens des courants) et telle quel la projection naturelle $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ donne un difféomorphisme CR entre S et $\widetilde{S}$. ***Pour citer cet article : P. Dolbeault et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit S une sous-variété réelle, lisse, compacte, de codimension 2 de $\mathbb{C}^n$, $n > 2$. On considère le problème de l'existence d'une hypersurface Levi-plate de bord S. Pour $n = 2$ ce problème a été étudié par plusieurs auteurs (cfr. [1,2,12,11,5,13]). Pour $n > 2$ le problème est surdéterminé. Une première condition nécessaire est que, au voisinage d'un point CR, S ne soit pas minimale au sens de Tumanov [14]. Soit maintenant $p \in S$ un point à tangence complexe. Alors, pour un choix convenable des coordonnées holomorphes locales $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, S est donnée par une équation $w = Q(z) + \mathrm{O}(\|z\|^3)$, où Q est une forme $\mathbb{R}$-bilinéaire, à valeurs complexes. On observe alors que si S est,

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localement au point p, le bord d'une hypersurface Levi-plate, des coordonnées (z, w) comme ci-dessus existent telles que la partie (1,1) de Q prend ses valeurs dans une droite réelle de $\mathbb{C}$. On dit plat un tel point et plat et elliptique si (z, w) peuvent être choisies de telle sorte que $Q(z)$ soit une forme réelle définie positive. Le résultat principal est contenu dans le théorème suivant :

Théorème 0.1. *Supposons que* (i) *S est non minimale en tout point CR*; (ii) *tout point complexe de S est plat et elliptique et il en existe un au moins*; (iii) *S ne contient aucune sous-variété complexe de dimension $n-2$. Alors il existe une sous-variété $\widetilde{M} \subset \mathbb{C} \times \mathbb{C}^n$, à singularités négligeables, avec bord $\widetilde{S}$ (au sens des courants) telle que $\widetilde{M} \setminus \widetilde{S}$ soit Levi-plate et la projection naturelle $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ donne un difféomorphisme CR entre S et $\widetilde{S}$.*

1. Introduction

Let S be a smooth 2-codimensional real compact submanifold of $\mathbb{C}^n$, $n > 2$. We address the problem of finding a Levi-flat hypersurface with boundary S. For $n = 2$ the problem was studied by several authors (see e.g. [1,2,12,11, 5,13]). The situation here turns out to be totally different from what it is in $\mathbb{C}^2$. The first difference is that a boundary of a hypersurface in general position is totally real in $\mathbb{C}^2$ but no more such in $\mathbb{C}^n$ with $n > 2$. Furthermore, if a surface $S \subset \mathbb{C}^2$ is real-analytic, any real-analytic foliation of S by real curves extends locally to a foliation of $\mathbb{C}^2$ by complex curves and hence S locally bounds many possible Levi-flat hypersurfaces. On the other hand, in higher dimension, a real-analytic submanifold $S \subset \mathbb{C}^n$ of codimension 2 in general position does not even locally bound a Levi-flat hypersurface M.

In Section 2 we study the necessary local compatibility conditions needed for a 2-codimensional smooth real submanifold $S \subset \mathbb{C}^n$ to bound a Levi-flat hypersurface at least locally. First we observe that near a CR point S must be nowhere minimal i.e. all local CR orbits must be of positive codimension. Next we consider a complex point $p \in S$ and local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, vanishing at p, such that S is locally given by the equation $w = Q(z) + \mathrm{O}(|z|^3)$, where $Q(z)$ is a complex valued quadratic form in the real coordinates $(\Re z, \Im z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ or, equivalently, in $(z, \bar{z})$. We observe that, if S bounds a Levi-flat hypersurface, Q has to satisfy a certain flatness condition, in which case we call p 'flat'. We further call a flat point $p \in S$ 'elliptic' if $Q(z) \in \mathbb{R}_+$ for every $z \neq 0$. A 2-codimensional submanifold $S \subset \mathbb{C}^n$ can only have isolated elliptic flat complex points.

In our main result, we take these local necessary conditions as our assumptions and obtain a global conclusion. There are no more global assumptions on S as it was the case in $\mathbb{C}^2$. There is the technical difficulty that the possible Levi-flat hypersurface solving the boundary problem may have self-intersections producing singularities even along large sets. We illustrate this phenomenon on the following example.

Example 1. Let $\mathbb{S}^{2n-2} \subset \mathbb{C}^{n-1} \times \mathbb{R} \cong \mathbb{R}^{2n-1}$ be the unit sphere and consider an immersion $\gamma : [-1, 1] \to \mathbb{C}$ with $\gamma(x) \neq \gamma(-x)$ for all $x \neq 0$. Then the image $S \subset \mathbb{C}^n$ of $\mathbb{S}^{2n-2}$ under the map $\mathrm{id} \times \gamma : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{C}^{n-1} \times \mathbb{C}$ is a submanifold of codimension 2 that locally bounds a Levi-flat hypersurface, image under the same map of open pieces of the unit ball $\mathbb{B}^{2n-1}$ bounded by $\mathbb{S}^{2n-2}$ in $\mathbb{C}^n$. The image of the ball itself can be singular at points $(z, \gamma(x))$ if there exist points $x' \neq x$ with $\gamma(x) = \gamma(x')$ whose set can be very large.

In view of this example we have to allow more general Levi-flat 'hypersurfaces' that are obtained as images of real manifolds. It is clear that already the complex leaves may have singularities away from their boundaries. Hence we are led to the following real analogue of complex-analytic varieties.

Let X be a complex manifold endowed with a Hermitian metric. Let $\mathcal{H}^d$ be the d-dimensional Hausdorff measure on X. A closed subset Y of X is said to be a *d-subvariety with negligible singularities*, of class C^k, $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, if there exists a closed subset σ of Y such that $\mathcal{H}^d(\sigma) = 0$ and $Y \setminus \sigma$ is an oriented real d-dimensional submanifold of class C^k of $X \setminus \sigma$ of locally finite $\mathcal{H}^d$ measure. The minimal set σ as above is called the *singular set* of Y (*scar set* according to Harvey and Lawson [9]) and $\operatorname{Reg} Y = Y \setminus \sigma$ its *regular part*. By integration on $\operatorname{Reg} Y$, we define a measurable, locally rectifiable current on X, denoted $[Y]$ and said to be the *integration current* on Y. The closed set σ may be increased without changing $[Y]$. A d-subvariety Y with negligible singularities is said to be CR, of CR dimension h and CR codimension m if there exists a closed subset $\sigma' \supset \sigma$ such that $\mathcal{H}^d(\sigma') = 0$ and $Y \setminus \sigma'$ is a CR submanifold of CR dimension h and CR codimension m. The classical definitions of CR geometry extend to d-subvarieties (and to currents [10]). A CR d-subvariety is said to be *Levi-flat* if its regular part is Levi-flat.

Finally, we add a local assumption on S guaranteeing that all the CR orbits have the same dimension and hence define a smooth foliation away from the complex points. We have:

Theorem 1.1. *Let $S \subset \mathbb{C}^n$, $n > 2$, be a compact connected smooth real 2-codimensional submanifold such that the following holds*:

(i) *S is nonminimal at every CR point*;
(ii) *every complex point of S is flat and elliptic and there exists at least one such point*;
(iii) *S does not contain complex submanifolds of dimension $(n-2)$.*

Then there exists a compact subvariety $\widetilde{M} \subset \mathbb{C} \times \mathbb{C}^n$ with boundary $\widetilde{S}$ (in the sense of currents) such that $\widetilde{M} \setminus \widetilde{S}$ is Levi-flat and the natural projection $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ restricts to a CR diffeomorphism between $\widetilde{S}$ and S.

2. Local analysis and flatness conditions

Given a smooth submanifold M of $\mathbb{C}^n$ we denote H_pM the complex tangent space to M at a point p.

Let S be a smooth real submanifold of real codimension 2 in $\mathbb{C}^n$ (not necessarily compact). We say that S is a *locally flat boundary* at a point $p \in M$ if an open neighbourhood of p in S locally bounds a Levi-flat hypersurface $M \subset \mathbb{C}^n$. Assume that S is a locally flat boundary and let $p \in S$ be such that S is *CR* near p. Then, near p, S is either a complex hypersurface (in which case it is clearly a locally flat boundary) or a generic submanifold of $\mathbb{C}^n$ at least at some points. In the second case being a locally flat boundary turns out to be a nontrivial condition for $n \geqslant 3$. Indeed, suppose that $M \subset \mathbb{C}^n$ is a Levi-flat hypersurface bounded by a generic submanifold S. Consider the foliation by complex hypersurfaces of M where it extends smoothly to the boundary. Since the boundary S is generic, it cannot be tangent to a complex leaf. Hence the leaf M_p of M through p intersects S transversally along a real hypersurface $S_p \subset S$. Since $H_qS \subset H_qM = T_qM_p$ for $q \in S_p$ near p, it follows that $H_qS_p = H_qS$ for such q. Hence S cannot be minimal (in the sense of Tumanov [14]) at p with p being arbitrary generic smooth boundary point. In fact, it follows that S_p is either a single *CR* orbit of S or a union of *CR* orbits.

When $n = 2$, S_p is totally real (i.e. $HS_p = \{0\}$) and hence is obviously nowhere minimal. If $n \geqslant 3$, S_p cannot be totally real for dimension reason and its 'nowhere minimality' becomes a nontrivial condition. $S \subset \mathbb{C}^n$ near a complex point $p \in S$ (i.e. such that T_pS is a complex hyperplane in $T_p\mathbb{C}^n$), for suitable holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ vanishing at p, is locally given by an equation

$$w = Q(z) + \mathrm{O}\big(|z|^3\big), \qquad Q(z) = \sum_{1 \leqslant i,j \leqslant n-1} (a_{ij} z_i z_j + b_{ij} z_i \bar{z}_j + c_{ij} \bar{z}_i \bar{z}_j), \tag{1}$$

where (a_{ij}) and (c_{ij}) are symmetric complex matrices and (b_{ij}) is an arbitrary complex matrix. The form $Q(z)$ can be seen as a "fundamental form" of S at p, however it is not uniquely determined (as a tensor too). A holomorphic quadratic change of coordinates of the form $(z, w) \mapsto (z, w + \sum a'_{ij} z_i z_j)$ results in adding (a'_{ij}) to the matrix (a_{ij}). It can be easily seen that the matrices (b_{ij}) and (c_{ij}) transform as tensors $T_pS \times T_pS \to T_p\mathbb{C}^n / T_pS$ under all biholomorphic changes of (z, w) preserving the form in (1).

2.1. A necessary condition at complex points

Clearly any symmetric $\mathbb{C}$-valued $\mathbb{R}$-bilinear form Q on $\mathbb{C}^n$ can appear in Eq. (1). However we shall see that the condition on S to be a locally flat boundary at p implies a nontrivial condition on Q.

We call S *flat* at a complex point $p \in S$ if, in some (and hence in any) coordinates (z, w) as in (1), there exists a complex number $\lambda \in \mathbb{C}$ such that $\sum b_{ij} z_i \bar{z}_j \in \lambda\mathbb{R}$ for all $z = (z_1, \ldots, z_{n-1})$.

If (b_{ij}) is a Hermitian matrix, S as above is automatically flat at p; in case $n = 2$ the flatness always holds. But any submanifold $S \subset \mathbb{C}^3_{z_1,z_2,w}$ given by $w = |z_1|^2 + \mathrm{i}|z_2|^2 + \mathrm{O}(|z|^3)$ is not flat at 0. Then:

Lemma 2.1. *Let $S \subset \mathbb{C}^n$ be a locally flat boundary with complex point $p \in S$. Then S is flat at p.*

If S is flat, by making a change of coordinates $(z,w)\mapsto(z,\lambda w)$, it is easy to make $\sum b_{ij}z_i\bar z_j\in\mathbb{R}$ for all z. Furthermore, by a change of coordinates $(z,w)\mapsto(z,w+\sum a'_{ij}z_iz_j)$ we can choose the holomorphic term in (1) to be the conjugate of the antiholomorphic one and so make the whole form Q real-valued. We now assume that S is nonminimal in its generic points and flat at complex points. We then prove that S is a locally flat boundary near p assuming in addition a certain positivity (ellipticity) condition. The latter condition is analogous to that for 2-surfaces in $\mathbb{C}^2$.

Let $p\in S$ be a flat point. We say that p is *elliptic* if, in some (and hence in any) coordinates (z,w), the quadratic form $Q(z)$ is real and positive definite. By adding $Q(z)$ and $Q(iz)$ we see that the ellipticity implies that the matrix (b_{ij}) is nonzero.

Remark 1. This definition generalizes the well-known notion of ellipticity, in the sense of Bishop [3], for $n=2$. Note that $S\subset\mathbb{C}^2$ is always flat at any complex point. In general, it can be shown that S is elliptic at a flat complex point p if and only if every intersection $L\cap S$ with a complex 2-plane L through p satisfying $L\not\subset T_pS$ is elliptic in $L\cong\mathbb{C}^2$ in the sense of Bishop.

The simplest example of S is the quadric of $\mathbb{C}^3$, $w=Q(z)$, where Q is as in (1). In our case when S is flat and elliptic at $p=0$, we can choose the coordinates (z,w) where $Q(z)$ is real and positive definite.

Lemma 2.2. *Suppose that the quadric $w=Q(z)$ is flat and elliptic at* 0. *Then it is CR and nowhere minimal outside* 0, *and the CR orbits are precisely the* 3*-dimensional ellipsoids given by $w=const$. The Levi form at the CR points is positive definite.*

In particular, it follows that elliptic flat points are always isolated complex points. This property also holds for general 2-codimensional submanifolds $S\subset\mathbb{C}^n$ as can be seen by comparing S with a corresponding approximating quadric.

For elliptic flat points, we now show that the above necessary conditions are in some sense sufficient for S to be a locally flat boundary.

Proposition 2.3. *Assume that $S\subset\mathbb{C}^n$ $(n\geqslant 3)$ is nowhere minimal at all its CR points and has an elliptic flat complex point p. Then a neighbourhood of p is foliated by compact real $(2n-3)$-dimensional CR orbits and there exists a Lipschitz function ν, smooth and without critical points away from p, having the CR orbits as the level surfaces.*

3. Global consequences of the local flatness

We now consider a compact real 2-codimensional submanifold S of $\mathbb{C}^n$, $n\geqslant 3$.

3.1. Induced foliation by CR orbits

We use classical topological tools to obtain a description of the global structure of the foliation.

Proposition 3.1. *Let $S\subset\mathbb{C}^n$ be a compact connected real* 2*-codimensional submanifold such that the conditions* (i)–(iii) *of Theorem* 1.1 *hold. Then S is homeomorphic to the unit sphere $\mathbb{S}^{2n-2}\subset\mathbb{C}^{n-1}_z\times\mathbb{R}_x$ such that the complex points are the poles $\{x=\pm1\}$ and the CR orbits in S correspond to the $(2n-3)$-spheres given by $x=const$. In particular, if S_{ell} denotes the (finite) set of all elliptic flat complex points of S, the open subset $S_0=S\setminus S_{\mathrm{ell}}$ carries a foliation $\mathcal{F}$ of class C^∞ with* 1*-codimensional compact leaves.*

The proof is based on Thurston's stability theorem (see e.g. [4], Theorem 6.2.1).

We now return to our central question: *When does a compact submanifold S of $\mathbb{C}^n$ bounds a Levi-flat hypersurface M?* From Proposition 3.1, we know that every CR orbit of S is a connected compact maximally complex CR submanifold of $\mathbb{C}^n$, $n\geqslant 3$, and hence, in view of the classical result of Harvey–Lawson [10], bounds a complex-analytic subvariety. Thus, in order to find M, at least as a real "subvariety", foliated by complex subvarieties, a natural way to proceed is to build it as a family of the solutions of the boundary problems for individual CR orbits. To do it, we reduce the problem to the corresponding problem in a real hyperplane of $\mathbb{C}^{n+1}$. The latter case is treated in the next section.

4. On boundaries of families of holomorphic chains with C^∞ parameters

Here we extend to the C^∞ case the C^ω solution of the boundary problem in a real hyperplane of $\mathbb{C}^n$ [7]. As in [7] we follow the method of Harvey–Lawson in [9], Section 3.

Notations: $n \geqslant 4$; $z'' = (z_2, \dots, z_{n-1}) \in \mathbb{C}^{n-2}$, $\zeta' = (x_1, z'') \in \mathbb{R} \times \mathbb{C}^{n-2}$. Let $E = \mathbb{R} \times \mathbb{C}^{n-1} = \{y_1 = 0\} \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$, and $k : E \to R_{x_1}$, $(x_1; z''; z_n) \mapsto x_1$. For $x_1^0 \in \mathbb{R}_{x_1}$, let $E_{x_1^0} = k^{-1}(x_1^0) = \{x_1 = x_1^0\}$.

Let $N \subset E$ be a compact, (oriented) CR subvariety of $\mathbb{C}^n$ of real dimension $2n-4$ and CR dimension $n-3$, ($n \geqslant 4$), of class C^∞, with negligible singularities (i.e. there exists a closed subset $\tau \subset N$ of $(2n-4)$-dimensional Hausdorff measure 0 such that $N \setminus \tau$ is a CR submanifold in $E \setminus \tau$). Assume that N, as a current of integration, is d-closed and satisfies:

(H) there exists a closed subset $\tau' \supset \tau$ of N with $\mathcal{H}^{2n-4}(\tau') = 0$ such that for every $z \in N \smallsetminus \tau'$, $N \smallsetminus \tau'$ is a submanifold transversal to the maximal complex affine subspace of E through z;

(H$'$) there exists a closed subset $L_0 \subset \mathbb{R}_{x_1}$ with $\mathcal{H}^1(L_0) = 0$ such that for every $x_0 \in k(N) \smallsetminus L_0$, the fiber $k^{-1}(x_0) \cap N$ is connected. We shall fix L_0 as above.

For every $x_1 \in k(N)$, let $N_{x_1} = N \cap E_{x_1}$ and consider the points $z \in N_{x_1}$ such that: (i) either $z \in \tau$; or (ii) E_{x_1} is not transverse to N at z.

Let τ'_{x_1} be the set of such points in N_{x_1} and $L := L_0 \cup \{x_1 \in \mathbb{R}\colon \mathcal{H}^{2n-5}(\tau'_{x_1}) > 0\}$. Clearly, $\mathcal{H}^1(L) = 0$.

Theorem 4.1. *Let N satisfy* (H) *and* (H$'$) *and L be chosen as above. Then, there exists, in $E' = E \setminus k^{-1}(L)$, a unique C^∞ maximally complex $(2n-3)$-subvariety M with negligible singularities in $E' \smallsetminus N$, foliated by complex $(n-2)$-subvarieties, with the properties that* supp $M \subset\subset E'$ *and M simply* (*or trivially*) *extends to E' by a $(2n-3)$-current* (*still denoted M*) *such that $dM = N$ in E'. The leaves are the sections by the hyperplanes $E_{x_1^0}$, $x_1^0 \in k(N) \setminus L$, and are the solutions of the "Harvey–Lawson problem" for finding a holomorphic chain in $E_{x_1^0} \cong \mathbb{C}^{n-1}$ with prescribed boundary $N \cap E_{x_1^0}$.*

In [7,8], the statement [7], Théorème 6.9 is given for $E' = E$, $n \geqslant 4$, and for N real analytic (C^ω), in two particular cases. The proof for C^∞ regularity, uses again [9], for any N with negligible singularities, but outside $k^{-1}(L)$. The C^ω hypothesis, and the particular cases we considered, allowed to go back to situations of Harvey–Lawson [10].

5. On some Levi-flat $(2n-1)$-subvarieties with given boundary in $\mathbb{C}^n$

We now return to the initial problem of finding a real Levi-flat hypersurface in $\mathbb{C}^n$ with prescribed boundary. We translate this problem into a boundary problem for subvarieties of a hyperplane E of $\mathbb{C}^{n+1}$ with negligible singularities, foliated by holomorphic varieties and then apply Theorem 4.1. We mention that Delannay [6] gives a solution of the problem under certain additional assumptions.

We first show that there exists a global Lipschitz function $\nu : S \to \mathbb{R}$ that is smooth and without critical points away from the complex points. By Proposition 2.1, such ν can be constructed near every complex point. Furthermore, in view of Proposition 3.1 such ν can be obtained globally on S away from its complex points. Putting everything together and using a partition of unity,we obtain a function $\nu : S \to \mathbb{R}$ with desired properties.

We now set $\widetilde{S} = N = \operatorname{gr} \nu = \{(\nu(z), z)\colon z \in S\}$. Then, as a consequence of the properties of ν and the description of the CR orbits in Proposition 3.1, all the assumptions of Theorem 4.1 are satisfied. We conclude that N is the boundary of a Levi-flat $(2n-2)$-variety $\widetilde{M}$ in $\mathbb{R} \times \mathbb{C}^n$. Taking $\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ to be the standard projection, we complete the proof of Theorem 1.1.

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