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Mathematical Problems in Mechanics/Partial Differential Equations

On the existence of solutions in weighted Sobolev spaces for the exterior Oseen problem

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Abstract

In this Note, we present existence and uniqueness results for the exterior Oseen problem. In order to control the behavior at infinity of functions, we use as functional framework weighted Sobolev spaces. The results rely on a L^p-theory for $1 < p < \infty$. ***To cite this article: C. Amrouche, U. Razafison, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Espaces de Sobolev avec poids pour le problème extérieur d'Oseen. Dans cette Note, on présente des résultats d'existence et d'unicité pour les équations d'Oseen posées dans des domaines extérieurs de $\mathbb{R}^3$. Le comportement à l'infini des solutions est décrit par l'utilisation des espaces de Sobolev avec poids. L'étude repose sur une théorie L^p, avec $1 < p < \infty$. ***Pour citer cet article : C. Amrouche, U. Razafison, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Dans un ouvert extérieur Ω, supposé connexe, on considère le problème d'Oseen (1), sous la condition à l'infini (2). Le système (1) est une linéarisation des équations de Navier–Stokes décrivant un écoulement de fluide autour d'un obstacle. Le but de cette Note est de présenter des résultats d'existence et d'unicité reposant sur une théorie L^p, avec $1 < p < \infty$ (voir [2] pour plus de details). Le domaine étant non borné, le choix des espaces de Sobolev avec poids comme cadre fonctionnel s'avère adéquat pour décrire le comportement à l'infini des solutions ainsi que de toutes ces dérivées successives. À notre connaissance, cette approche a été utilisée par Finn [6] et Farwig [4,5]. Notons que dans [4] et [2], l'étude repose aussi sur des espaces avec poids anisotropes, afin de tenir compte de la région, appelée *sillage*, apparaîssant derrière l'obstacle durant l'écoulement. On se limitera ici au cas des poids isotropes. Parmi les nombreux autres travaux sur les équations d'Oseen, on peut citer ceux de Galdi [7,8] où les espaces de Sobolev homogènes sont considérés.

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Les Théorèmes 3.2 et 3.3 constituent les principaux résultats de cette Note. En particulier, le Théorème 3.2 montre que, sous des hypothèses minimales sur les données $\boldsymbol{f}$, g et $\boldsymbol{u}_*$, le problème (1) admet un champs de vitesses $\boldsymbol{u}$ satisfaisant $\nabla\boldsymbol{u} \in L^p(\Omega)$, pour tout $1 < p < \infty$. Pour obtenir l'existence de tels champs de vitesses, l'espace $Z_p(\Omega)$ pour la donnée g est optimal. Lorsque $p > 3/2$, Galdi (voir [8, Théorèmes VII.7.2 et VII.7.3]) montre l'existence de telles solutions pour $g \in W_0^{-1,p}(\Omega) \cap L^p(\Omega)$. Ce dernier espace étant strictement inclus dans $Z_p(\Omega)$. Par ailleurs, le résultat du Théorème 3.2 pour le cas $1 < p \leqslant 3/2$ est, à notre connaissance, nouveau. La démonstration des deux théorèmes repose sur les propriétés du problème d'Oseen dans un domaine borné et dans $\mathbb{R}^3$ qui sont établies dans [1].

1. Introduction

Let $\mathcal{B}$ be a bounded open set of $\mathbb{R}^3$, not necessarily connected, with a Lipschitz-continuous boundary Γ and let Ω denotes the complement of $\overline{\mathcal{B}}$. We assume that Ω is connected. Consider the following exterior Oseen problem: find a pair $(\boldsymbol{u}, \pi)$ satisfying :

$$\begin{aligned} &-\Delta\boldsymbol{u} + \frac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi = \boldsymbol{f} \quad \text{in } \Omega,\\ &\operatorname{div}\boldsymbol{u} = g \quad \text{in } \Omega,\\ &\boldsymbol{u} = \boldsymbol{u}_* \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where the data are the external forces $\boldsymbol{f}$ acting on the fluid, the velocity $\boldsymbol{u}_*$ of the fluid on the boundary Γ and a function g. The system above is a linearized approximation of the Navier–Stokes equations describing the flow of a fluid past the body $\overline{\mathcal{B}}$. The condition at infinity is supposed by

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_\infty, \tag{2}$$

where $\boldsymbol{u}_\infty$ is a given constant vector of $\mathbb{R}^3$. The aim of this Note is to present the existence and uniqueness results of Problem (1) proved in a previous work [2] and which rely on a L^p-theory for $1 < p < \infty$. Since the domain Ω is unbounded, the use of weighted Sobolev spaces is convenient to describe the behavior at infinity of the solutions and all their derivatives. To our knowledge, this approach for the study of (1), goes back to Finn [6], Farwig [4,5]. Observe that in [4] and [2], anisotropic weighted spaces are also considered in order to take into account the *wake region* which appears behind the body during the flow. Further investigations on the exterior Oseen problem are due, for instance, to Galdi [7,8] where homogeneous Sobolev spaces are used.

2. Notation and weighted Sobolev spaces

Before stating the main results, at this point, we shall explain the notation and we shall present the functional framework. Constant vectors and vector-valued functions are denoted with boldface characters. For any real $p \in]1, \infty[$, we define p' such that $1/p + 1/p' = 1$. A point of $\mathbb{R}^3$ is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$ and its distance to the origin $r = |\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We denote by $[k]$ the integer part of k. For any $j \in \mathbb{Z}$, $\mathbb{P}_j$ stands for the space of polynomials of degree less than or equal to j. We denote by $\mathcal{D}(\Omega)$ the space of $\mathcal{C}^\infty$ functions with compact support in Ω and $\mathcal{D}'(\Omega)$ is the space of distributions defined on Ω. For $m \geqslant 1$, we recall that $W^{m,p}(\Omega)$ is the well known classical Sobolev space. We shall write $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$ to mean $u \in W^{m,p}(\Omega')$, for any bounded domain Ω', with $\overline{\Omega'} \subset \Omega$.

Let us now introduce the weight function $\rho = 1 + r$ and the logarithmic weight function $\lg r = \ln(1 + \rho)$. For a nonnegative integer m and $\alpha \in \mathbb{R}$, we set

$$k = k(m, p, \alpha) = \begin{cases} -1 & \text{if } 3/p + \alpha \notin \{1, \dots, m\},\\ m - 3/p - \alpha & \text{if } 3/p + \alpha \in \{1, \dots, m\} \end{cases}$$

and we define the weighted Sobolev space

$$\begin{aligned} W^{m,p}_\alpha(\Omega) = \big\{ &u \in \mathcal{D}'(\Omega);\ \forall\boldsymbol{\lambda} \in \mathbb{N}^3,\ 0 \leqslant |\boldsymbol{\lambda}| \leqslant k,\ \rho^{\alpha-m+|\boldsymbol{\lambda}|}(\lg r)^{-1}\partial^{\boldsymbol{\lambda}} u \in L^p(\Omega),\ k+1 \leqslant |\boldsymbol{\lambda}| \leqslant m,\\ &\rho^{\alpha-m+|\lambda|}\partial^{\boldsymbol{\lambda}} u \in L^p(\Omega)\big\}, \end{aligned}$$

which is a Banach space equipped with its natural norm given by

$$\|u\|_{W^{m,p}_\alpha(\Omega)} = \left(\sum_{0\leqslant|\boldsymbol{\lambda}|\leqslant k} \left\| \rho^{\alpha-m+|\boldsymbol{\lambda}|} (\lg r)^{-1} \partial^{\boldsymbol{\lambda}} u \right\|^p_{L^p(\Omega)} + \sum_{k+1\leqslant|\boldsymbol{\lambda}|\leqslant m} \left\| \rho^{\alpha-m+|\lambda|} \partial^{\boldsymbol{\lambda}} u \right\|^p_{L^p(\Omega)} \right)^{1/p}.$$

For more details about the properties of these spaces, the reader can refer to [3]. Observe that all the local properties of the space $W^{m,p}_\alpha(\Omega)$ coincide with those of the Sobolev space $W^{m,p}(\Omega)$. As a consequence, the traces of these functions on Γ satisfy the usual trace theorems. This allows to define the space

$$\mathring{W}^{m,p}_\alpha(\Omega) = \left\{ u \in W^{m,p}_\alpha(\Omega),\ \gamma_0 u = \gamma_1 u = \cdots = \gamma_{m-1} u = 0 \right\}.$$

The space $\mathcal{D}(\Omega)$ is dense in $\mathring{W}^{m,p}_\alpha(\Omega)$ and therefore, its dual space, $W^{-m,p'}_{-\alpha}(\Omega)$ is a space of distributions. We now define the spaces

$$\tilde{L}^p(\mathbb{R}^3) = \left\{ g \in L^p(\mathbb{R}^3),\ \frac{\partial g}{\partial x_1} \in W^{-2,p}_0(\mathbb{R}^3) \right\} \quad \text{and} \quad \widetilde{W}^{1,p}_0(\Omega) = \left\{ u \in W^{1,p}_0(\Omega),\ \frac{\partial u}{\partial x_1} \in W^{-1,p}_0(\Omega) \right\},$$

which are Banach spaces equipped with the norms of the graph. Observe that the space $\mathcal{D}(\overline{\Omega})$ is dense in $\widetilde{W}^{1,p}_0(\Omega)$. We define the space

$$\mathring{\widetilde{W}}{}^{1,p}_0(\Omega) = \left\{ u \in \widetilde{W}^{1,p}_0(\Omega),\ u = 0 \text{ on } \Gamma \right\}.$$

We also introduce the following space

$$Z_p(\Omega) = \left\{ \tilde{g}|_\Omega,\ \tilde{g} \in \tilde{L}^p(\mathbb{R}^3),\ \forall \lambda \in \mathbb{P}_{[2-3/p']},\ \left\langle \frac{\partial \tilde{g}}{\partial x_1}, \lambda \right\rangle_{W^{-2,p}_0(\mathbb{R}^3)\times W^{2,p'}_0(\mathbb{R}^3)} = 0 \right\}$$

which is a Banach space for the norm

$$\|g\|_{Z_p(\Omega)} = \inf\left\{ \|\tilde{g}\|_{\tilde{L}^p(\mathbb{R}^3)};\ \tilde{g}|_\Omega = g,\ \tilde{g} \in \tilde{L}^p(\mathbb{R}^3),\ \forall \lambda \in \mathbb{P}_{[2-3/p']},\ \left\langle \frac{\partial \tilde{g}}{\partial x_1}, \lambda \right\rangle_{W^{-2,p}_0(\mathbb{R}^3)\times W^{2,p'}_0(\mathbb{R}^3)} = 0 \right\}.$$

3. Main results

In what follows, we shall assume that Ω is an exterior domain of $\mathbb{R}^3$ with a $\mathcal{C}^{1,1}$ boundary.

Definition 3.1. Let $1 < p < \infty$ be given. Let $\gamma, \delta \in \mathbb{R}$ be such that $\gamma \in [3,4]$, $\gamma > p$ and $\delta \in [3/2, 2]$ and $\delta > p$. We define two reals $r_1 = r(p,\delta)$ and $r_2 = r_2(p,\gamma)$ as follow:

$$\frac{1}{r_1} = \frac{1}{p} - \frac{1}{\delta} \quad \text{and} \quad \frac{1}{r_2} = \frac{1}{p} - \frac{1}{\gamma}.$$

For instance, the particular case $p = 3/2$, then r_1 ranges $[3, \infty[$ and r_2 ranges $[12/5, 3]$.

For any pair $(\boldsymbol{u}, \pi)$, we define the operator $\mathbf{T}(\boldsymbol{u}, \pi) = (-\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi, -\operatorname{div} \boldsymbol{u})$ and we introduce the kernel of the operator $\mathbf{T}$, denoted by $\mathcal{N}^+_p(\Omega)$ and defined as follow:

$$\mathcal{N}^+_p(\Omega) = \left\{ (\boldsymbol{u}, \pi) \in \mathring{\widetilde{W}}{}^{1,p}_0(\Omega) \times L^p(\Omega),\ \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega \right\}.$$

Moreover, if $1 < p < 4$, we assume in addition $\boldsymbol{u} \in L^{r_2}(\Omega)$. By the same way, we denote by $\mathcal{N}^-_p(\Omega)$ the kernel of $\mathbf{T}^*$, where $\mathbf{T}^* = (-\Delta \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi, -\operatorname{div} \boldsymbol{u})$. These spaces can be characterized (see [2, Proposition 5.14]). In Particular, if $1 < p < 4$, then we have $\mathcal{N}^+_p(\Omega) = \mathcal{N}^-_p(\Omega) = \{(\mathbf{0}, 0)\}$. We now state the following:

Theorem 3.2. *Let $\boldsymbol{f} \in W^{-1,p}_0(\Omega)$, $g \in Z_p(\Omega)$ and $\boldsymbol{u}_* \in W^{1/p',p}(\Gamma)$ satisfy the following compatibility condition, for $1 < p \leqslant 4/3$,*

$$\forall (\boldsymbol{v}, \eta) \in \mathcal{N}^-_{p'}(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \langle g, \eta \rangle + \left\langle (\nabla \boldsymbol{v} - \eta I).\boldsymbol{n}, \boldsymbol{u}_* \right\rangle_{W^{-1/p',p'}(\Gamma)\times W^{1/p',p}(\Gamma)} = 0,$$

where I is the second order identity tensor.

(i) *If* $1 < p < 4$, *then Problem* (1) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\widetilde{W}_0^{1,p}(\Omega) \cap L^{r_2}(\Omega)) \times L^p(\Omega)$ *satisfying the estimate*

$$\|\boldsymbol{u}\|_{L^{r_2}(\Omega)} + \|\boldsymbol{u}\|_{\widetilde{W}_0^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \leqslant C\big(\|\boldsymbol{f}\|_{W_0^{-1,p}(\Omega)} + \|g\|_{Z_p(\Omega)} + \|\boldsymbol{u}_*\|_{W^{1/p',p}(\Gamma)}\big).$$

(ii) *If* $p \geqslant 4$, *then Problem* (1) *has a solution* $(\boldsymbol{u}, \pi) \in \widetilde{W}_0^{1,p}(\Omega) \times L^p(\Omega)$, *unique up to an element of* $\mathcal{N}_p^+(\Omega)$, *satisfying the estimate*

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}_p^+(\Omega)} \big(\|\boldsymbol{u}+\boldsymbol{v}\|_{\widetilde{W}_0^{1,p}(\Omega)} + \|\pi+\eta\|_{L^p(\Omega)}\big) \leqslant C\big(\|\boldsymbol{f}\|_{W_0^{-1,p}(\Omega)} + \|g\|_{Z_p(\Omega)} + \|\boldsymbol{u}_*\|_{W^{1/p',p}(\Gamma)}\big).$$

Proof. The detailed proof of this theorem can be found in [2] and it is made of three steps.

(i) *The case* $p = 2$. For $g = 0$ and $\boldsymbol{u}_* = \boldsymbol{0}$, we easily establish the existence of a weak solution $(\boldsymbol{u}, \pi)$ that belongs to $\mathring{W}_0^{1,2}(\Omega) \times L^2_{\mathrm{loc}}(\overline{\Omega})$. Then combining existence results on the Oseen equations in $\mathbb{R}^3$ [1, Theorem 3.6] and in bounded domains [2, Proposition 4.6], we prove that the pressure π belongs to $L^2(\Omega)$, the velocity $\boldsymbol{u}$ also belongs to $L^4(\Omega)$ and the solution $(\boldsymbol{u}, \pi)$ is unique. The case $\boldsymbol{u}_* \neq \boldsymbol{0}$ is proved by lifting $\boldsymbol{u}_*$ by a field $\boldsymbol{w} \in H^1(\Omega)$ such that $\operatorname{div} \boldsymbol{w} = 0$ in Ω. To solve the complete problem, we first show that the data $\boldsymbol{f}$ and g can respectively be extending by $\tilde{\boldsymbol{f}} \in W_0^{-1,2}(\mathbb{R}^3)$ and $\tilde{g} \in \tilde{L}^2(\mathbb{R}^3)$ satisfying the condition $\langle \frac{\partial \tilde{g}}{\partial x_1}, 1\rangle_{W_0^{-2,2}(\mathbb{R}^3)\times W_0^{-2,2}(\mathbb{R}^3)} = 0$. The resolution of the problem in the whole space $\mathbb{R}^3$ with data $\tilde{\boldsymbol{f}}$ and $\tilde{g}$ finally allows to bring back to the case $\boldsymbol{f} = \boldsymbol{0}$ and $g = 0$ in Ω.

(ii) *The case* $p > 2$. Assume first $\boldsymbol{u}_* = \boldsymbol{0}$ and the data $\boldsymbol{f}$ and g have compact supports. It follows that $\boldsymbol{f} \in W_0^{-1,2}(\Omega)$ and $g \in Z_2(\Omega)$. Then we verify that the solution $(\boldsymbol{u}, \pi) \in \mathring{\widetilde{W}}_0^{1,2}(\Omega) \times L^2(\Omega)$, given by (i), also belongs to $(\mathring{\widetilde{W}}_0^{1,p}(\Omega) \cap L^{r_2}(\Omega)) \times L^p(\Omega)$. Thanks to this result, we can prove that for any $\boldsymbol{u}_* \in W^{1/p',p}(\Gamma)$, Problem (1) with $\boldsymbol{f} = \boldsymbol{0}$ and $g = 0$, has a unique solution $(\boldsymbol{u}, \pi) \in (\widetilde{W}_0^{1,p}(\Omega) \cap \widetilde{W}_0^{1,2}(\Omega)) \times (L^p(\Omega) \cap L^2(\Omega))$. Therefore we can assume $\boldsymbol{u}_* = \boldsymbol{0}$ and if now $\boldsymbol{f}$ and g do not have compact supports, we can construct extensions satisfying $\tilde{\boldsymbol{f}} \in W_0^{-1,p}(\mathbb{R}^3)$ and $\tilde{g} \in \tilde{L}^p(\mathbb{R}^3)$ such that, for any $\lambda \in \mathbb{P}_{[2-3/p']}$, $\langle \frac{\partial \tilde{g}}{\partial x_1}, \lambda\rangle_{W_0^{-2,p}(\mathbb{R}^3)\times W_0^{2,p'}(\mathbb{R}^3)} = 0$. Using the properties of the Oseen problem in $\mathbb{R}^3$, we prove the existence and the uniqueness of the pair $(\boldsymbol{u}, \pi) \in (\mathring{\widetilde{W}}_0^{1,p}(\Omega) \cap L^{r_2}(\Omega)) \times L^p(\Omega)$ satisfying (1). The case $\boldsymbol{u}_* \neq \boldsymbol{0}$ is solved by lifting $\boldsymbol{u}_*$ by a field $\boldsymbol{w} \in \widetilde{W}_0^{1,p}(\Omega)$ such that $\operatorname{div} \boldsymbol{w} = 0$ in Ω. We then proceed as in the first step (i).

(iii) *The case* $1 < p < 2$. We use a duality argument. Given $\boldsymbol{f} \in W_0^{-1,p}(\Omega)$ satisfying the compatibility condition: for any $(\boldsymbol{w}, \tau) \in \mathcal{N}_{p'}^-(\Omega)$, $\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)} = 0$ if $1 < p \leqslant 4/3$, we establish the existence and the uniqueness of $\boldsymbol{u} \in \mathring{W}_0^{1,p}(\Omega)$ satisfying: for any $(\boldsymbol{v}, q) \in \mathring{\widetilde{W}}_0^{1,p'}(\Omega) \times L^{p'}(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = 0$,

$$\left\langle \boldsymbol{u}, -\Delta \boldsymbol{v} - \frac{\partial \boldsymbol{v}}{\partial x_1} + \nabla q \right\rangle_{\mathring{W}_0^{1,p}(\Omega)\times W_0^{-1,p}(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{W_0^{-1,p}(\Omega)\times \mathring{W}_0^{1,p'}(\Omega)}.$$

Next, as in the first step (i), we prove the existence of a pressure $\pi \in L^p(\Omega)$ satisfying (1), for data $\boldsymbol{u}_* = \boldsymbol{0}$ and $g = 0$. Finally we proceed as previously to solve the complete problem. □

Remark 1. If $p > 3/2$, Galdi proved a similar result for data $g \in W_0^{-1,p}(\Omega) \cap L^p(\Omega)$ (see [8, Theorems VII.7.2 and VII.7.3]). Observe that if $p > 3/2$, $W_0^{-1,p}(\Omega) \cap L^p(\Omega) \subset Z_p(\Omega)$. Hence, the previous theorem improves the one proved by Galdi. Besides, to our knowledge, the case $1 < p \leqslant 3/2$ is a new result.

Remark 2. If $1 < p < 3$, then the velocity $\boldsymbol{u}$ given by Theorem 3.2 tends to $\boldsymbol{0}$ at infinity in the following sense,

$$\lim_{|\boldsymbol{x}|\to\infty} \int_{S_2} \big|\boldsymbol{u}\big(\sigma|\boldsymbol{x}|\big) - \boldsymbol{u}_\infty\big| \,\mathrm{d}\sigma = 0, \tag{1}$$

where S_2 is the unit sphere of $\mathbb{R}^3$. If $3 < p < 4$, then $\boldsymbol{u}$ tends to $\mathbf{0}$ pointwise. As a consequence, given a nonzero constant vector $\boldsymbol{u}_\infty \in \mathbb{R}^3$ and $(\boldsymbol{f}, g, \boldsymbol{u}_*)$ satisfying the assumptions of the theorem, then Problem (1) has a unique solution $(\boldsymbol{u}, \pi) \in \mathcal{D}'(\Omega) \times L^p(\Omega)$ satisfying $\nabla \boldsymbol{u} \in L^p(\Omega)$, $\frac{\partial \boldsymbol{u}}{\partial x_1} \in W_0^{-1,p}(\Omega)$ and $\boldsymbol{u}$ tends to $\boldsymbol{u}_\infty$ in the sense of (1), for $1 < p < 3$, and pointwise for $3 < p < 4$.

Let us now introduce the kernel $\mathcal{A}_p^+(\Omega)$ defined by:

$$\mathcal{A}_p^+(\Omega) = \left\{(\boldsymbol{u}, \pi) \in W_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega),\ \frac{\partial \boldsymbol{u}}{\partial x_1} \in L^p(\Omega), \mathbf{T}(\boldsymbol{u}, \pi) = (\mathbf{0}, 0) \text{ in } \Omega,\ \boldsymbol{u} = \mathbf{0} \text{ on } \partial\Omega\right\}.$$

Moreover, if $1 < p < 4$, then we assume that $\nabla \boldsymbol{u}$ belongs $L^{r_2}(\Omega)$ and, if $1 < p < 2$, $\boldsymbol{u}$ belongs to $L^{r_1}(\Omega)$. From Theorem 3.2 and the properties of the Oseen problem in $\mathbb{R}^3$ (see [1, Theorem 3.2]), we derive the second main result.

Theorem 3.3. *Let Ω be an exterior domain with a $\mathcal{C}^{1,1}$ boundary. Given $(\boldsymbol{f}, g) \in L^p(\Omega) \times \widetilde{W}_0^{1,p}(\Omega)$ and $\boldsymbol{u}_* \in W^{1+1/p',p}(\partial\Omega)$, then*

(i) *If $1 < p < 2$, Problem (1) has a unique solution $(\boldsymbol{u}, \pi) \in L^{r_1}(\Omega) \times W_0^{1,p}(\Omega)$ such that $\nabla \boldsymbol{u} \in L^{r_2}(\Omega)$, $\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \in L^p(\Omega)$ and $\frac{\partial \boldsymbol{u}}{\partial x_1} \in L^p(\Omega)$. Moreover we have*

$$\|\boldsymbol{u}\|_{L^{r_1}(\Omega)} + \|\nabla \boldsymbol{u}\|_{L^{r_2}(\Omega)} + \left\|\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j}\right\|_{L^p(\Omega)} + \left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_{L^p(\Omega)} + \|\pi\|_{W_0^{1,p}(\Omega)}$$
$$\leqslant C\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|g\|_{\widetilde{W}_0^{1,p}(\Omega)} + \|\boldsymbol{u}_*\|_{W^{1+1/p',p}(\partial\Omega)}\big).$$

(ii) *If $2 \leqslant p < 3$, Problem (1) has a solution $(\boldsymbol{u}, \pi) \in W_0^{1,r_2}(\Omega) \times W_0^{1,p}(\Omega)$, unique up to an element of $\mathcal{A}_p^+(\Omega)$, such that $\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \in L^p(\Omega)$ and $\frac{\partial \boldsymbol{u}}{\partial x_1} \in L^p(\Omega)$. Furthermore, we have*

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{A}_p^+(\Omega)} \big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W_0^{1,r_2}(\Omega)} + \|\pi + \mu\|_{W_0^{1,p}(\Omega)}\big) + \left\|\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j}\right\|_{L^p(\Omega)} + \left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_{L^p(\Omega)} \tag{2}$$
$$\leqslant C\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|g\|_{\widetilde{W}_0^{1,p}(\Omega)} + \|\boldsymbol{u}_*\|_{W^{1+1/p',p}(\partial\Omega)}\big). \tag{3}$$

(iii) *If $p \geqslant 3$, Problem (1) has a solution $(\boldsymbol{u}, \pi) \in W_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$, unique up to an element of $\mathcal{A}_p^+(\Omega)$, such that $\frac{\partial \boldsymbol{u}}{\partial x_1} \in L^p(\Omega)$ and satisfying*

$$\inf_{(\boldsymbol{\lambda}, \mu) \in \mathcal{A}_p^+(\Omega)} \big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{W_0^{2,p}(\Omega)} + \|\pi + \mu\|_{W_0^{1,p}(\Omega)}\big) + \left\|\frac{\partial \boldsymbol{u}}{\partial x_1}\right\|_{L^p(\Omega)}$$
$$\leqslant C\big(\|\boldsymbol{f}\|_{L^p(\Omega)} + \|g\|_{\widetilde{W}_0^{1,p}(\Omega)} + \|\boldsymbol{u}_*\|_{W^{1+1/p',p}(\partial\Omega)}\big).$$

Remark 3. (i) If $1 < p \leqslant 3/2$, then the velocity $\boldsymbol{u}$ given by the previous theorem tends $\mathbf{0}$ at infinity in the sense of (1). Hence, as for Remark 2, given $\boldsymbol{u}_\infty \in \mathbb{R}^3$ and $(\boldsymbol{f}, g, \boldsymbol{u}_*)$ satisfying the assumptions of the previous theorem, one can prove that Problem (1) has a unique solution $(\boldsymbol{u}, \pi) \in \mathcal{D}'(\Omega) \times W_0^{1,p}(\Omega)$ such that $\nabla \boldsymbol{u} \in L^{r_2}(\Omega)$ and $\lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_\infty$.

(ii) If $3/2 < p < 2$, then $\boldsymbol{u}$ is continuous and tends to $\mathbf{0}$ pointwise. We can therefore have a similar conclusion as above.

(iii) Observe that if $1 < p < 3$, then $\nabla \boldsymbol{u}$ and π tend to zero at infinity in the sense of Definition 3.1.

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