



Differential Geometry

# A refinement of the Ray–Singer torsion

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## Abstract

We propose a refinement of the Ray–Singer torsion, which can be viewed as an analytic counterpart of the refined combinatorial torsion introduced by Turaev. Given a closed, oriented manifold of odd dimension with fundamental group  $\Gamma$ , the refined torsion is a complex valued, holomorphic function defined for representations of  $\Gamma$  which are close to the space of unitary representations. When the representation is unitary the absolute value of the refined torsion is equal to the Ray–Singer torsion, while its phase is determined by the  $\eta$ -invariant. As an application we extend and improve a result of Farber about the relationship between the absolute torsion of Farber–Turaev and the  $\eta$ -invariant. **To cite this article:** M. Braverman, T. Kappeler, *C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

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## Résumé

**Raffinement de la torsion de Ray–Singer.** Nous proposons un raffinement de la torsion analytique de Ray–Singer, qui peut être considéré comme un équivalent analytique du raffinement de la torsion combinatoire introduit par Turaev. Soit  $M$  une variété fermée et orientée de dimension impaire et de groupe fondamental  $\Gamma$ . La torsion analytique raffinée est une fonction holomorphe à valeurs complexes, définie pour les représentations de  $\Gamma$ , qui sont proches de l'espace des représentations unitaires. Dans le cas où la représentation est unitaire, la valeur absolue de la torsion analytique raffinée est égale à la torsion de Ray–Singer dès que sa phase est déterminée par l'invariant  $\eta$ . Comme application, nous généralisons et améliorons un résultat de Farber concernant la relation entre la torsion absolue de Farber–Turaev et l'invariant  $\eta$ . **Pour citer cet article :** M. Braverman, T. Kappeler, *C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

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## Version française abrégée

Soit  $E \rightarrow M$  un fibré complexe plat où  $M$  est une variété fermée et orientée de dimension impaire. Pour un ensemble ouvert de connexions  $\nabla$  plates et acycliques de  $E$ , qui contient toutes les connexions hermitiennes et acycliques, nous proposons un raffinement de la torsion analytique de Ray–Singer  $T^{\text{RS}}(\nabla)$ . Dès que  $T^{\text{RS}}(\nabla)$  est un

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nombre positive réel,  $T = T(\nabla)$  est dans le cas général un nombre complexe et, en conséquence, a une phase non-triviale. Ce raffinement peut être considéré comme un équivalent analytique du raffinement du concept de la torsion combinatoire introduit par Turaev [13,14] et développé plus tard par Farber et Turaev [9,10]. Bien que, en général,  $T$  ne soit pas égal à la torsion de Turaev, les deux torsions sont étroitement liées.

Si  $\dim M \equiv 1 \pmod{4}$  ou si le rang du fibré  $E$  est divisible par 4, la torsion analytique raffinée  $T = T(\nabla)$  est indépendante de tous les choix faits pour la définir. Si  $\dim M \equiv 3 \pmod{4}$ ,  $T(\nabla)$  dépend du choix d'une variété compacte et orientée  $N$  ; le bord orienté de celle-ci est difféomorphe à deux copies disjointes de  $M$ , mais seulement à un facteur  $i^{k \cdot \text{rk} E}$  ( $k \in \mathbb{Z}$ ) près.

Si la connexion  $\nabla$  est hermitienne, c'est-à-dire s'il existe une métrique hermitienne pour  $E$  qui est invariante par  $\nabla$ , alors la torsion analytique raffinée  $T$  est un nombre complexe dont la valeur absolue est égale à la torsion de Ray–Singer et dont la phase est déterminée par l'invariant  $\eta$  de l'opérateur de signature. Si  $\nabla$  n'est pas hermitienne, les relations entre la torsion analytique raffinée, la torsion de Ray–Singer et l'invariant  $\eta$  sont un peu plus compliquées, cf. Section 6.

Une des propriétés les plus importantes de la torsion analytique raffinée est qu'elle dépend, dans un sens approprié, d'une manière *holomorphe* de la connexion  $\nabla$ . Le fait qu'il soit possible d'utiliser la torsion de Ray–Singer et l'invariant  $\eta$  pour définir une fonction holomorphe permet d'appliquer les méthodes de l'analyse complexe pour l'étude des deux invariants. En particulier, nous obtenons une relation entre la torsion analytique raffinée et la torsion combinatoire raffinée de Turaev qui généralise le célèbre théorème de Cheeger–Müller concernant l'égalité entre la torsion analytique de Ray–Singer et la torsion combinatoire [7,12]. Comme application, nous généralisons et améliorons un résultat de Farber relatif à la relation entre la torsion absolue de Farber–Turaev et l'invariant  $\eta$ .

Notre construction de la torsion analytique raffinée utilise le superdéterminant régularisé de  $\zeta$  de l'opérateur de signature. On remarquera que cet opérateur n'est pas autoadjoint.

L'idée de représenter la torsion de Ray–Singer comme valeur absolue d'une fonction holomorphe définie sur l'espace des connexions est due à Burghlea et Haller [5]. En réponse à une version préliminaire de notre article, Dan Burghlea nous a signalé son projet (mené avec Haller) [6] de construction d'une fonction holomorphe utilisant des déterminants régularisés de certains opérateurs non autoadjoints du type d'un opérateur Laplacien.

## 1. The odd signature operator

Let  $M$  be a closed oriented manifold of odd dimension  $\dim M = n = 2r - 1$  and let  $E$  be a complex vector bundle over  $M$  endowed with a flat connection  $\nabla$ . Let  $\Omega^\bullet(M, E)$  denote the space of smooth differential forms on  $M$  with values in  $E$  and set

$$\Omega^{\text{even}}(M, E) = \bigoplus_{p=0}^{r-1} \Omega^{2p}(M, E).$$

Fix a Riemannian metric  $g^M$  on  $M$  and let  $*$ :  $\Omega^\bullet(M, E) \rightarrow \Omega^{n-\bullet}(M, E)$  denote the Hodge  $*$ -operator. The (even part of) the *odd signature operator* is the operator  $B_{\text{even}} = B_{\text{even}}(\nabla, g^M): \Omega^{\text{even}}(M, E) \rightarrow \Omega^{\text{even}}(M, E)$ , whose value on a form  $\omega \in \Omega^{2p}(M, E)$  is defined by the formula

$$B_{\text{even}}\omega := i^r (-1)^{p+1} (*\nabla - \nabla*)\omega \in \Omega^{n-2p-1}(M, E) \oplus \Omega^{n-2p+1}(M, E).$$

The odd signature operator was introduced by Atiyah, Patodi, and Singer, [1, p. 44], [2, p. 405], and, in the more general setting used here, by Gilkey [11, pp. 64–65].

The operator  $B_{\text{even}}$  is an elliptic differential operator, whose leading symbol is symmetric with respect to any Hermitian metric  $h^E$  on  $E$ .

In this Note we define the refined analytic torsion in the case when the pair  $(\nabla, g^M)$  satisfies the following simplifying assumptions. The general case will be addressed elsewhere.

**Assumption 1.1.** The connection  $\nabla$  is acyclic, i.e.,

$$\text{Im}(\nabla|_{\Omega^{k-1}(M, E)}) = \text{Ker}(\nabla|_{\Omega^k(M, E)}), \quad \text{for every } k = 0, \dots, n.$$

**Assumption 1.2.** The odd signature operator  $B_{\text{even}} = B_{\text{even}}(\nabla, g^M)$  is bijective.

Note that all acyclic Hermitian connections satisfy Assumptions 1.1 and 1.2. By a simple continuity argument, these two assumptions are then satisfied for all flat connections in an open neighborhood (in  $C^0$ -topology) of the set of acyclic Hermitian connections.

**2. Graded determinant**

Set

$$\Omega_+^k(M, E) := \text{Ker}(\nabla*) \cap \Omega^k(M, E), \quad \Omega_-^k(M, E) := \text{Ker}(*\nabla) \cap \Omega^k(M, E). \tag{1}$$

Assumption 1.2 implies that  $\Omega^k(M, E) = \Omega_+^k(M, E) \oplus \Omega_-^k(M, E)$ . Hence, 1 defines a *grading* on  $\Omega^k(M, E)$ .

Define  $\Omega_{\pm}^{\text{even}}(M, E) = \bigoplus_{p=0}^{r-1} \Omega_{\pm}^{2p}(M, E)$  and let  $B_{\text{even}}^{\pm}$  denote the restriction of  $B_{\text{even}}$  to  $\Omega_{\pm}^{\text{even}}(M, E)$ . It is easy to see that  $B_{\text{even}}$  leaves the subspaces  $\Omega_{\pm}^{\text{even}}(M, E)$  invariant and it follows from Assumption 1.2 that the operators  $B_{\text{even}}^{\pm} : \Omega_{\pm}^{\text{even}}(M, E) \rightarrow \Omega_{\pm}^{\text{even}}(M, E)$  are bijective.

One of the central objects of this paper is the *graded determinant* of the operator  $B_{\text{even}}$ . To construct it we need to choose a *spectral cut* along a ray  $R_{\theta} = \{\rho e^{i\theta} : 0 \leq \rho < \infty\}$ , where  $\theta \in [-\pi, \pi)$  is an Agmon angle for  $B_{\text{even}}$ . Since the leading symbol of  $B_{\text{even}}$  is symmetric,  $B_{\text{even}}$  admits an Agmon angle  $\theta \in (-\pi, 0)$ . Given such an angle  $\theta$ , observe that it is an Agmon angle for  $B_{\text{even}}^{\pm}$  as well. The graded determinant of  $B_{\text{even}}$  is the non-zero complex number defined by the formula

$$\text{Det}_{\text{gr},\theta}(B_{\text{even}}) := \frac{\text{Det}_{\theta}(B_{\text{even}}^+)}{\text{Det}_{\theta}(B_{\text{even}}^-)}. \tag{2}$$

By standard arguments,  $\text{Det}_{\text{gr},\theta}(B_{\text{even}})$  is independent of the choice of the Agmon angle  $\theta \in (-\pi, 0)$ .

**3. A convenient choice of the Agmon angle**

For  $\mathcal{I} \subset \mathbb{R}$  we denote by  $L_{\mathcal{I}}$  the solid angle

$$L_{\mathcal{I}} = \{\rho e^{i\theta} : 0 < \rho < \infty, \theta \in \mathcal{I}\}.$$

Though many of our results are valid for any Agmon angle  $\theta \in (-\pi, 0)$ , some of them are easier formulated if the following conditions are satisfied:

(AG1)  $\theta \in (-\pi/2, 0)$ , and

(AG2) there are no eigenvalues of the operator  $B_{\text{even}}$  in the solid angles  $L_{(-\pi/2,\theta]}$  and  $L_{(\pi/2,\theta+\pi]}$ .

For the sake of simplicity of exposition, we will assume that  $\theta$  is chosen so that these conditions are satisfied. Since the leading symbol of  $B_{\text{even}}$  is symmetric, such a choice of  $\theta$  is always possible.

**4. Relationship with the Ray–Singer torsion and the  $\eta$ -invariant**

For a pair  $(\nabla, g^M)$  satisfying Assumptions 1.1 and 1.2 set

$$\xi = \xi(\nabla, g^M, \theta) := \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \zeta'_{2\theta}(0, (-1)^{k+1} (*\nabla)^2|_{\Omega_+^k(M,E)}), \tag{3}$$

where  $\zeta'_{2\theta}(s, (-1)^{k+1} (*\nabla)^2|_{\Omega_+^k(M,E)})$  is the derivative with respect to  $s$  of the  $\zeta$ -function of the operator  $(*\nabla)^2|_{\Omega_+^k(M,E)}$  corresponding to the spectral cut along the ray  $R_{2\theta}$ , and  $\theta$  is an Agmon angle satisfying (AG1)–(AG2).

Let  $\eta = \eta(\nabla, g^M)$  denote the  $\eta$ -invariant of the operator  $B_{\text{even}}(\nabla, g^M)$ , cf. Definition 4.2 of [4]. Note that, since the operator  $B(\nabla, g^{TM})$  is not, in general, self-adjoint,  $\eta$  might be a complex number, cf. [11] and Section 4 of [4]. Theorem 7.2 of [4] implies that,

$$\text{Det}_{\text{gr},\theta}(B_{\text{even}}) = e^{\xi(\nabla, g^M, \theta)} \cdot e^{-i\pi\eta(\nabla, g^M)}. \tag{4}$$

This representation of the graded determinant turns out to be very useful, e.g., in computing the metric anomaly of  $\text{Det}_{\text{gr},\theta}(B_{\text{even}})$ .

If the connection  $\nabla$  is Hermitian, then (3) coincides with the well-known expression for the logarithm of the Ray–Singer torsion  $T^{\text{RS}} = T^{\text{RS}}(\nabla)$ . Hence, for a Hermitian connection  $\nabla$  we have

$$\xi(\nabla, g^M, \theta) = \log T^{\text{RS}}(\nabla).$$

If  $\nabla$  is not Hermitian but is sufficiently close (in  $C^0$ -topology) to an acyclic Hermitian connection, then Theorem 8.2 of [4] states that

$$\log T^{\text{RS}}(\nabla) = \text{Re } \xi(\nabla, g^M, \theta). \quad (5)$$

Combining (5) and (4), we get

$$|\text{Det}_{\text{gr},\theta}(B_{\text{even}})| = T^{\text{RS}}(\nabla) \cdot e^{\pi \text{Im } \eta(\nabla, g^M)}. \quad (6)$$

If  $\nabla$  is Hermitian, then the operator  $B_{\text{even}}$  is self-adjoint and  $\eta = \eta(\nabla, g^M)$  is real. Hence, for the case of an acyclic Hermitian connection we obtain from (6)

$$|\text{Det}_{\text{gr},\theta}(B_{\text{even}})| = T^{\text{RS}}(\nabla).$$

## 5. Definition of the refined analytic torsion

The graded determinant of the odd signature operator is not a differential invariant of the connection  $\nabla$  since, in general, it depends on the choice of the Riemannian metric  $g^M$ . Using the metric anomaly of the  $\eta$ -invariant computed by Gilkey, [11], we investigate in §9 of [4] the *metric anomaly* of the graded determinant and then use it to ‘correct’ the graded determinant and construct a differential invariant—the refined analytic torsion.

Suppose an acyclic connection  $\nabla$  is given. We call a Riemannian metric  $g^M$  on  $M$  *admissible* for  $\nabla$  if the operator  $B_{\text{even}} = B_{\text{even}}(\nabla, g^M)$  satisfies Assumption 1.2. We denote the set of admissible metrics by  $\mathcal{B}(\nabla)$ . The set  $\mathcal{B}(\nabla)$  might be empty. However, admissible metrics exist for all flat connections in an open neighborhood (in  $C^0$ -topology) of the set of acyclic Hermitian connections, cf. Proposition 6.8 of [4].

**Definition 5.1.** The *refined analytic torsion*  $T(\nabla)$  corresponding to an acyclic connection  $\nabla$ , satisfying  $\mathcal{B}(\nabla) \neq \emptyset$ , is defined as follows: fix an admissible Riemannian metric  $g^M \in \mathcal{B}(\nabla)$  and let  $\theta \in (-\pi, 0)$  be an Agmon angle for  $B_{\text{even}}(\nabla, g^M)$ .

(i) If  $\dim M \equiv 1 \pmod{4}$  then

$$T(\nabla) = T(M, E, \nabla) := \text{Det}_{\text{gr},\theta}(B_{\text{even}}(\nabla, g^M)) \in \mathbb{C} \setminus 0.$$

(ii) If  $\dim M \equiv 3 \pmod{4}$  choose a smooth compact oriented manifold  $N$  whose oriented boundary is diffeomorphic to two disjoint copies of  $M$  (since  $\dim M$  is odd, such a manifold always exists, cf. [15]). Then

$$T(\nabla) = T(M, E, \nabla, N) := \text{Det}_{\text{gr},\theta}(B_{\text{even}}) \cdot \exp\left(i\pi \frac{\text{rank } E}{2} \int_N L(p)\right) \in \mathbb{C} \setminus 0,$$

where  $L(p)$  is the Hirzebruch  $L$ -polynomial in the Pontrjagin forms of a Riemannian metric on  $N$  which is a product near  $M$ .

Note that  $\int_N L(p)$  is real and, hence,  $|T(\nabla)| = |\text{Det}_{\text{gr},\theta}(B_{\text{even}})|$ .

If  $\nabla$  is close enough to an acyclic Hermitian connection, then  $\mathcal{B}(\nabla) \neq \emptyset$  and it is shown in §§11–12 of [4], that  $T(\nabla)$  is independent of the choices of the admissible metric  $g^M$  and the Agmon angle  $\theta \in (-\pi, 0)$ . However, if  $\dim M \equiv 3 \pmod{4}$ , then the refined analytic torsion *does depend on the choice of the manifold*  $N$ . The quotient of the refined torsions corresponding to different choices of  $N$  is a complex number of the form  $i^{k \cdot \text{rank } E}$  ( $k \in \mathbb{Z}$ ). Hence, if  $\text{rank } E$  is even then  $T(\nabla)$  is well defined up to a sign, and if  $\text{rank } E$  is divisible by 4, then  $T(\nabla)$  is a well defined complex number.

### 6. Comparison with the Ray–Singer torsion

The equality (6) implies that, if  $\nabla$  is  $C^0$ -close to an acyclic Hermitian connection, then

$$\log \frac{|T(\nabla)|}{T^{\text{RS}}(\nabla)} = \pi \operatorname{Im} \eta(\nabla, g^M) \tag{7}$$

In particular, if  $\nabla$  is an acyclic Hermitian connection, then  $|T(\nabla)| = T^{\text{RS}}(\nabla)$ .

Let  $\mathbf{Arg}_\nabla$  denote the unique cohomology class  $\mathbf{Arg}_\nabla \in H^1(M, \mathbb{C}/\mathbb{Z})$  such that for every closed curve  $\gamma \in M$  we have  $\det(\operatorname{Mon}_\nabla(\gamma)) = \exp(2\pi i \langle \mathbf{Arg}_\nabla, [\gamma] \rangle)$ , where  $\operatorname{Mon}_\nabla(\gamma)$  denotes the monodromy of the flat connection  $\nabla$  along the curve  $\gamma$  and  $\langle \cdot, \cdot \rangle$  denotes the natural pairing  $H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}/\mathbb{Z}$ . Then, cf. Theorem 12.8 of [4], if  $\nabla$  is  $C^0$ -close to an acyclic Hermitian connection, then

$$\log \frac{|T(\nabla)|}{T^{\text{RS}}(\nabla)} = \pi (\langle L(p) \cup \operatorname{Im} \mathbf{Arg}_\nabla, [M] \rangle). \tag{8}$$

If  $\dim M \equiv 3 \pmod{4}$ , then  $L(p)$  has no component of degree  $\dim M - 1$  and, hence,  $|T(\nabla)| = T^{\text{RS}}(\nabla)$ .

### 7. The refined analytic torsion as a holomorphic function on the space of representations

One of the main properties of the refined analytic torsion  $T(\nabla)$  is that, in an appropriate sense, it depends holomorphically on the connection. Note, however, that the space of connections is infinite dimensional and one needs to choose an appropriate notion of a holomorphic function on such a space. As an alternative one can view the refined analytic torsion as a holomorphic function on a finite-dimensional space, which we shall now explain.

The set  $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$  of all  $n$ -dimensional complex representations of  $\pi_1(M)$  has a natural structure of a complex algebraic variety. Each representation  $\alpha \in \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$  gives rise to a vector bundle  $E_\alpha$  with a flat connection  $\nabla_\alpha$ , whose monodromy is isomorphic to  $\alpha$ . Let  $\operatorname{Rep}_0(\pi_1(M), \mathbb{C}^n) \subset \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$  denote the set of all representations  $\alpha \in \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$  such that the connection  $\nabla_\alpha$  is acyclic. Further we denote by  $\operatorname{Rep}^u(\pi_1(M), \mathbb{C}^n) \subset \operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$  the set of all unitary representations and set  $\operatorname{Rep}_0^u(\pi_1(M), \mathbb{C}^n) = \operatorname{Rep}^u(\pi_1(M), \mathbb{C}^n) \cap \operatorname{Rep}_0(\pi_1(M), \mathbb{C}^n)$ .

Denote by  $V \subset \operatorname{Rep}_0(\pi_1(M), \mathbb{C}^n)$  the set of representations  $\alpha$  for which there exists a metric  $g^M$  so that the odd signature operator  $B_{\text{even}}(\nabla, g^M)$  is bijective (i.e., Assumption 1.2 is satisfied). It is easy to see that  $V$  is an open neighborhood of the set  $\operatorname{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$  of acyclic unitary representations.

For every  $\alpha \in V$  one defines the refined analytic torsion  $T_\alpha := T(\nabla_\alpha)$ . Corollary 13.11 of [4] states that the function  $\alpha \mapsto T_\alpha$  is holomorphic on the open set of all non-singular points of  $V$ .

### 8. Comparison with Turaev’s torsion

In [13,14], Turaev introduced a refinement  $T_\alpha^{\text{comb}}(\varepsilon, \sigma)$  of the combinatorial torsion associated to a representation  $\alpha$  of  $\pi_1(M)$ . This refinement depends on an additional combinatorial data, denoted by  $\varepsilon$  and called the *Euler structure* as well as on the *cohomological orientation* of  $M$ , i.e., on the orientation  $\sigma$  of the determinant line of the cohomology  $H^*(M, \mathbb{R})$  of  $M$ . There are two versions of the Turaev torsion—the homological and the cohomological one. It is more convenient for us to use the cohomological Turaev torsion as it is defined in Section 9.2 of [10]. For  $\alpha \in \operatorname{Rep}_0(\pi_1(M), \mathbb{C}^n)$  the cohomological Turaev torsion  $T_\alpha^{\text{comb}}(\varepsilon, \sigma)$  is a non-vanishing complex number.

Theorem 10.2 of [10] computes the quotient of the Turaev and the Ray–Singer torsions. Combined with (8) this leads to the following result: for every Euler structure  $\varepsilon$  and every cohomological orientation  $\sigma$ , there exists an open neighborhood  $V' \subset V$  of  $\operatorname{Rep}_0^u(\pi_1(M), \mathbb{C}^n)$  such that for every  $\alpha \in V'$

$$\left| \frac{T_\alpha}{T_\alpha^{\text{comb}}(\varepsilon, \sigma)} \right| = |f_{\varepsilon, \sigma}(\alpha)|, \tag{9}$$

where  $f_{\varepsilon, \sigma}(\alpha)$  is a holomorphic function of  $\alpha \in V'$ , which is explicitly calculated in §14.4 of [4].

Let  $\Sigma$  denote the set of singular points of the complex analytic set  $\operatorname{Rep}(\pi_1(M), \mathbb{C}^n)$ . The refined analytic torsion  $T_\alpha$  is a non-vanishing holomorphic function of  $\alpha \in V \setminus \Sigma$ . By the very construction [13,14,10] the Turaev torsion is a non-vanishing holomorphic function of  $\alpha \in \operatorname{Rep}_0(\pi_1(M), \mathbb{C}^n)$ . Hence,  $(T_\alpha/T_\alpha^{\text{comb}})^2$  is a holomorphic function on  $V \setminus \Sigma$ . If

the absolute values of two non-vanishing holomorphic functions are equal on a connected open set then the functions must be equal up to a factor  $\phi \in \mathbb{C}$  with  $|\phi| = 1$ . This observation and (9) lead to the following generalization of the Cheeger–Müller theorem, cf. Theorem 14.5 of [4]: *For each connected component  $C$  of  $V'$ , there exists a constant  $\phi_C = \phi_C(\varepsilon, \mathfrak{o}) \in \mathbb{R}$ , depending on  $\varepsilon$  and  $\mathfrak{o}$ , such that*

$$\frac{T_\alpha}{T_\alpha^{\text{comb}}(\varepsilon, \mathfrak{o})} = e^{i\phi_C} f_{\varepsilon, \mathfrak{o}}(\alpha). \quad (10)$$

It would be interesting to extend (10) to a generalization of the Bismut–Zhang extension [3] of the Cheeger–Müller theorem.

## 9. Application: phase of the Turaev torsion

The equality (10) provides, in particular, a relationships between the phases of the Turaev torsion and the refined analytic torsion. If  $\alpha$  is a unitary representation, then (4) and Definition 5.1 express the phase of the refined analytic torsion in terms of the  $\eta$ -invariant. Thus we obtain a relationship between the phase of the Turaev torsion and the  $\eta$ -invariant for the case of a unitary representation  $\alpha$ , see Theorem 14.9 of [4] for the precise form of this relationship. In particular, this leads to an extension and a new proof of a theorem of Farber [8] about the relationship between the sign of the absolute torsion of Farber–Turaev, [9], and the  $\eta$ -invariant, cf. Theorem 14.12 of [9].

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