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Mathematical Physics

Feynman path integrals for the time dependent quartic oscillator

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Abstract

The Schrödinger equation with a time dependence in both a quadratic and a quartic potential is considered. Existence of solutions is shown and a rigorous Feynman path integral representation for the solution is given in terms of well-defined infinite-dimensional oscillatory integrals. ***To cite this article: S. Albeverio, S. Mazzucchi, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Intégrales de chemins de Feynman pour un oscillateur quartique dépendant du temps. On étudie une équation de Schrödinger avec une dépendence temporelle dans un potentiel quadratique ainsi que dans un potentiel quartique. L'existence de solutions est démontrée ainsi qu'une représentation en termes d'integrales de chemins de Feynman, définis rigoureusement comme intégrales oscillatoires en dimension infinie. ***Pour citer cet article : S. Albeverio, S. Mazzucchi, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

The present Note concerns the rigorous mathematical realization for the Feynman path integral representation of the solution of the Schrödinger equation, describing the time evolution of the state $\psi \in L^2(\mathbb{R}^d)$ of a d-dimensional non-relativistic quantum particle

$$\begin{cases} i\hbar \frac{\partial}{\partial t}\psi = -\dfrac{\hbar^2}{2m}\Delta\psi + V\psi, \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

(where $m > 0$ is the mass of the particle, $\hbar$ is the reduced Planck constant, $t \geqslant 0$, $x \in \mathbb{R}^d$). We consider the case where the potential V depends explicitly on the time variable t and is the sum of an harmonic oscillator part plus a quartic perturbation:

$$V(t) = \frac{1}{2}x\Omega^2(t)x + \alpha\lambda(t)C(x,x,x,x), \tag{2}$$

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where Ω and λ are respectively C^1 maps from the interval $[0,t]$ to the $d \times d$ symmetric positive matrices and $\mathbb{R}^+$, while C is a completely symmetric positive fourth order covariant tensor on $\mathbb{R}^d$ and α a positive constant.

In 1942 R.P. Feynman proposed a heuristic representation for the solution of Schrödinger equation (1). According to Feynman's alternative formulation of quantum mechanics, the state of the system at time t is given by an integral over the space of paths γ with fixed end point:

$$\psi(t,x) = \left(\int e^{\frac{i}{\hbar}S_t(\gamma)} D\gamma\right)^{-1} \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)} \psi_0\big(\gamma(0)\big) D\gamma \tag{3}$$

(where $S_t(\gamma) = \frac{m}{2}\int_0^t |\dot\gamma(s)|^2\,ds - \int_0^t V(s,\gamma(s))\,ds$ is the classical action of the system evaluated along the path γ and $D\gamma$ an heuristic "flat" measure). The study of rigorous mathematical definitions of the heuristic formula (3) begun in the 1960s and nowadays several approaches can be found in the physical and in the mathematical literature. The potentials which could be handled mathematically for a long time were only essentially of the type "harmonic oscillator plus perturbations which are Fourier (or Laplace) transforms of measures". In [2] the situation was changed, allowing effectively polynomially growing potentials. In the present Note we further extend these results to allow time dependent potentials of the type (2) by means of infinite-dimensional oscillatory integrals with polynomially growing phase function, a well defined class of functional integrals recently developed in [1,2]. (For a general discussion of time dependent potentials and references see e.g. [3].) The first step is the construction and the study of the family of evolution operators associated to Eq. (1). In the following we shall assume for notation simplicity that $m = 1$, but the whole discussion can be generalized to arbitrary values of the mass m. Let us denote by $H(t)$ the symmetric linear operator on $L^2(\mathbb{R}^d)$ given on $C_0^\infty(\mathbb{R}^d)$ by

$$H(t) = -\frac{\hbar}{2}\Delta + V(t), \tag{4}$$

where the potential $V(t)$ is given by (2).

Proposition 1. *Let $z \in \mathbb{C}$ be a complex parameter with* $\mathrm{Re}(z) \leqslant 0$. *Then, under the assumptions above on the potential V in* (2)*, the operators $H(t)$ have a common domain D and there exists a two parameter family of operators $U^z(t,s): L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $s,t \in \mathbb{R}$, such that*

- $U^z(r,s)U^z(s,t) = U^z(r,t)$;
- $U^z(t,t) = 1$;
- $U^z(t,s)$ *is jointly strongly continuous in t and s;*
- *if $\psi \in D$, then $\phi_s^z(t) := U^z(t,s)\psi$ is in D for all t and satisfies*

$$\frac{d}{dt}\phi_s^z(t) = zH(t)\phi_s^z(t), \qquad \phi_s^z(s) = \psi,$$

and $\|\phi_s^z(t)\| \leqslant \|\psi\|$ for all $t \geqslant s$.

Moreover if $\mathrm{Re}(z) = 0$, *then the operators $U^z(t,s)$ are unitary.*

Proof. First of all one has to prove that the time-dependent Hamiltonian operators $H(t)$ in (4) have a common domain D. Then, under the regularity assumptions on the maps Ω and λ, it is simple to see that for each $\psi \in L^2(\mathbb{R}^d)$, $(t-s)^{-1}(H(t)H(s)^{-1} - I)\psi$ is uniformly strongly continuous and uniformly bounded in s and t lying in any fixed compact subinterval of $\mathbb{R}$. Moreover $A(t)\psi := \lim_{s\uparrow t}(t-s)^{-1}(H(t)H(s)^{-1} - I)\psi$ exists uniformly for s,t in each compact subinterval and $A(t)$ is bounded and strongly continuous in t. The final result follows from Theorem X.70 in [4]. □

Let $\mathcal{D}$ be the subset of the complex plane given by $\mathcal{D} = \{z \in \mathbb{C}, \mathrm{Re}(z) < 0\}$ and let $\bar{\mathcal{D}}$ be its closure. For any $\psi, \phi \in D(H(t)) = D$, let us define the function $f: \bar{\mathcal{D}} \to \mathbb{C}$ given by

$$f(z) = \langle \psi, U^z(t,s)\phi \rangle.$$

The following holds:

Proposition 2. *f is analytic on $\mathcal{D}$ and continuous on $\bar{\mathcal{D}}$.*

Let us consider two vectors $\phi, \psi_0 \in L^2(\mathbb{R}^d)$ that are Fourier transforms of complex bounded variation measures on $\mathbb{R}^d$. More precisely let μ_0 be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_0 = \psi_0$ (with $\hat{}$ denoting Fourier transform). Let μ_ϕ be the complex bounded variation measure on $\mathbb{R}^d$ such that $\hat{\mu}_\phi(x) = (2\pi i\hbar)^{d/2} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x)$. Under suitable growth conditions on μ_0, μ_ϕ and if the time t is sufficiently small it is possible to give a well-defined mathematical meaning to the Feynman path integral representation of the weak solution of (1)

$$\big(\phi, \psi(t)\big) = \int_{\mathbb{R}^d} \bar{\phi}(x) \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar} S_t(\gamma)} \psi_0\big(\gamma(0)\big) D\gamma \, dx \tag{5}$$

as the analytic continuation (in the parameter α) of an infinite-dimensional generalized oscillatory integral on a suitable Hilbert space (see [2,1]). Let us consider the Hilbert space $\mathcal{H} = \mathbb{R}^d \times H_t$, where H_t is the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$, with $\gamma(0) = 0$ and inner product $\langle \gamma_1, \gamma_2 \rangle = \int_0^t \dot{\gamma}_1(s)\dot{\gamma}_2(s)\, ds$. Let us consider the operator $L : \mathcal{H} \to \mathcal{H}$ given by:

$$\begin{aligned}
&(x,\gamma) \to (y,\eta) = L(x,\gamma),\\
&y = \int_0^t \Omega^2(t-s)x \, ds + \int_0^t \Omega^2(t-s)\gamma(s)\, ds,\\
&\eta(s) = -\int_0^s \int_t^u \Omega^2(t-r)x \, dr\, du - \int_0^s \int_t^u \Omega^2(t-r)\gamma(r)\, dr\, du,
\end{aligned} \tag{6}$$

and the fourth order tensor operator B given by:

$$\begin{aligned}
&B\big((x_1,\gamma_1),(x_2,\gamma_2),(x_3,\gamma_3),(x_4,\gamma_4)\big)\\
&\quad = \int_0^t \lambda(t-s) C\big(\gamma_1(s)+x_1, \gamma_2(s)+x_2, \gamma_3(s)+x_3, \gamma_4(s)+x_4\big)\, ds.
\end{aligned} \tag{7}$$

Let us denote by $\bar{\Omega}$ the following quantity

$$\bar{\Omega} = \max_{s\in[0,t]} \big\|\bar{\Omega}(t-s)\big\|.$$

The following holds:

Theorem 3. *Let us assume that the following inequalities are satisfied*

$$\bar{\Omega} t < \frac{\pi}{2}, \qquad 1 - \bar{\Omega}\tan\big(\bar{\Omega} t\big) > 0. \tag{8}$$

Let us assume in addition that the measures μ_0, μ_ϕ satisfy the following assumption:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{\hbar}{4} x \bar{\Omega}^{-1} \tan(\bar{\Omega} t) x} e^{(y+\cos(\bar{\Omega} t)^{-1}x)(1-\bar{\Omega}\tan(\bar{\Omega} t))^{-1}(y+\cos(\bar{\Omega} t)^{-1}x)} |\mu_0|(dx)|\mu_\phi|(dy) < \infty. \tag{9}$$

Then, if $\alpha \leqslant 0$ the infinite-dimensional oscillatory integral

$$(2\pi i\hbar)^{d/2} \widetilde{\int_{\mathcal{H}}} e^{\frac{i}{2\hbar}(|x|^2+|\gamma|^2)} e^{-\frac{i}{2\hbar}\langle (x,\gamma), L(x,\gamma)\rangle} e^{-\frac{i\alpha}{\hbar} B((x,\gamma),(x,\gamma),(x,\gamma),(x,\gamma))} e^{-\frac{i}{2\hbar}|x|^2} \bar{\phi}(x)\psi_0\big(\gamma(t)+x\big)\, dx\, d\gamma \tag{10}$$

is well defined and is equal to the following absolutely convergent integral

$$(\mathrm{i}\hbar)^{d/2} \int\limits_{\mathbb{R}^d \times C_t} \mathrm{e}^{\mathrm{i}\alpha\hbar \int_0^t \lambda(t-s) C(\omega(s)+x,\omega(s)+x,\omega(s)+x,\omega(s)+x)\,\mathrm{d}s}$$
$$\times\, \mathrm{e}^{\frac{1}{2}\int_0^t (\omega(s)+x)\Omega^2(t-s)(\omega(s)+x)\,\mathrm{d}s}\, \bar{\phi}\big(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\hbar}x\big)\psi_0\big(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\hbar}\omega(t) + \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{\hbar}x\big) W(\mathrm{d}\omega)\,\mathrm{d}x \tag{11}$$

(*C_t being the space of continuous paths, $C_t = \{\omega \in C([0,t];\mathbb{R}^d) \mid \gamma(0) = 0\}$, and W the Wiener measure on it*).

Moreover, if $\alpha \geqslant 0$, the analytic continuation in α of the integral (11) *represents the scalar product between ϕ and the solution of the Schrödinger equation with initial datum ψ_0 and Hamiltonian* (4).

Proof. The first statement is an application of the theory in [2]. The second statement follows by Proposition 2 and the Feynman–Kac formula for the solution of the heat equation. □

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