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Numerical Analysis

Numerical analysis of the generalized von Kármán equations

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Abstract

The ‘generalized von Kármán equations’ constitute a mathematical model for a nonlinearly elastic plate subjected to boundary conditions ‘of von Kármán type’ only on a portion of its lateral face, the remaining portion being free. We establish here the convergence of a conforming finite element approximation to these equations. The proof relies in particular on a compactness method due to J.-L. Lions and on Brouwer’s fixed point theorem. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Analyse numérique des équations de von Kármán généralisées. Les « équations de von Kármán généralisées » constituent un modèle mathématique d’une plaque non linéairement élastique soumise à des conditions aux limites « de von Kármán » sur une partie seulement de sa face latérale, la partie restante étant libre. On établit ici la convergence de la solution approchés de ces équations, obtenue par une méthode d’élements finis conformes. La démonstration repose en particulier sur une méthode de compacité due à J.-L. Lions et sur le théorème du point fixe de Brouwer. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

1. The generalized von Kármán equations

Greek indices, corresponding to the coordinates in the ‘horizontal’ plane, vary in {1,2} and Latin indices vary in {1,2,3}, except if they are used for indexing sequences. The summation convention with respect to repeated indices is systematically used.

Let there be given a bounded, connected, simply-connected, open subset ω of the ‘horizontal’ plane $\mathbb{R}^2$ with a sufficiently smooth boundary γ, the set ω being locally on a single side of γ. Let γ_1 and γ_2 be two disjoint relatively open subsets of γ such that *length* $\gamma_1 > 0$, *length* $\gamma_2 > 0$, and *length* $(\gamma - \{\gamma_1 \cup \gamma_2\}) = 0$. Let $y = (y_\alpha)$ denote a generic point in $\bar{\omega}$, and let $\partial_\alpha = \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} = \partial^2/\partial y_\alpha \partial y_\beta$. Let (ν_α) denote the unit outer normal vector along γ,

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let (τ_α) denote the unit tangent vector along γ defined by $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$, and finally, let $\partial_\nu = \nu_\alpha \partial_\alpha$ and $\partial_\tau = \tau_\alpha \partial_\alpha$ denote the outer normal and tangential derivative operators along γ.

Consider a *nonlinearly elastic plate*, with *middle surface* $\bar{\omega}$ and *thickness* 2ε, whose constituting material is *homogeneous* and *isotropic*, and whose reference configuration $\bar{\omega} \times [-\varepsilon, \varepsilon]$ is a *natural state*. The behavior of this material is thus governed by its two Lamé constants $\lambda > 0$ and $\mu > 0$. The plate is subjected to *vertical body forces* in its interior and to *vertical surface forces* on its upper and lower faces. On the portion $\gamma_1 \times [-\varepsilon, \varepsilon]$ of its lateral face, the plate is subjected to *horizontal forces 'of von Kármán's type'*, of the form introduced in [2]. Finally, the plate is subjected to a boundary condition of *free edge* on the remaining portion $\gamma_2 \times [-\varepsilon, \varepsilon]$ of its lateral face.

As shown in [4], the leading term of a formal asymptotic expansion of the three-dimensional displacement field inside the plate, with the thickness as the 'small' parameter, can be fully computed from the solution of a *two-dimensional 'displacement' boundary value problem* posed over ω. The main result of [4] then consisted in showing that, under the assumption that the set ω is simply connected, there is a one-to-one correspondence between the smooth solutions of this boundary value problem and those of another boundary value problem, which takes the form of the following *generalized von Kármán equations*:

$$
\begin{aligned}
-\partial_{\alpha\beta} m_{\alpha\beta}(\nabla^2\xi) &= [\phi, \xi] + f \quad \text{in } \omega,\\
\Delta^2\phi &= -[\xi, \xi] \quad \text{in } \omega,\\
\xi = \partial_\nu \xi &= 0 \quad \text{on } \gamma_1,\\
m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta &= 0 \quad \text{on } \gamma_2,\\
\partial_\alpha m_{\alpha\beta}(\nabla^2\xi)(y)\nu_\beta + \partial_\tau\big(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta\big) &= 0 \quad \text{on } \gamma_2,\\
\phi = \phi_0 \quad \text{and} \quad \partial_\nu\phi &= \phi_1 \quad \text{on } \gamma.
\end{aligned}
$$

Up to appropriate multiplicative constants, the *two unknowns* $\xi : \bar{\omega} \to \mathbb{R}$ and $\phi : \bar{\omega} \to \mathbb{R}$ represent the *vertical component of the displacement field of the middle surface* $\bar{\omega}$ *of the plate* and the *Airy function*. The *Monge–Ampère form* $[.,.]$ is defined for smooth enough functions $\phi : \bar{\omega} \to \mathbb{R}$ and $\xi : \bar{\omega} \to \mathbb{R}$ by $[\phi, \xi] = \partial_{11}\phi\partial_{22}\xi + \partial_{22}\phi\partial_{11}\xi - 2\partial_{12}\phi\partial_{12}\xi$. The given function $f \in L^2(\omega)$ takes into account the vertical forces. The functions $m_{\alpha\beta}(\nabla^2\xi)$ are defined for a smooth enough function $\xi : \bar{\omega} \to \mathbb{R}$ by

$$
m_{\alpha\beta}(\nabla^2\xi) := -\frac{1}{3} a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\xi, \quad \text{where } a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}).
$$

Finally, the functions $\phi_0 \in H^{3/2}(\gamma)$ and $\phi_1 \in H^{1/2}(\gamma)$ are given. These 'generalized' von Kármán equations generalize the 'classical' von Kármán equations that correspond to the special case where $\gamma_1 = \gamma$. Detailed treatments of these classical equations are found in [3] and [7].

Let the bilinear mapping $B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega)$ be defined as follows: Given $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B(\xi, \eta) \in H_0^2(\omega)$ is the unique solution of the variational equations (note that $[\xi, \eta] \in L^1(\omega)$):

$$
\int_\omega \Delta B(\xi, \eta)\Delta\theta \,\mathrm{d}\omega = \int_\omega [\xi, \eta]\theta \,\mathrm{d}\omega \quad \text{for all } \theta \in H_0^2(\omega).
$$

Define another bilinear mapping

$$
\widetilde{B} : H^2(\omega) \times H^2(\omega) \to V(\omega) := \big\{\eta \in H^2(\omega);\ \eta = \partial_\nu\eta = 0 \text{ on } \gamma_1\big\}
$$

as follows: Given $(\phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\widetilde{B}(\phi, \xi) \in V(\omega)$ is the unique solution of the variational equations:

$$
\big(\big(\widetilde{B}(\phi, \xi), \eta\big)\big) = \int_\omega [\phi, \xi]\eta \,\mathrm{d}\omega \quad \text{for all } \eta \in V(\omega),
$$

where $((\cdot,\cdot))$ is the inner-product on $V(\omega)$ defined by

$$
((\zeta, \eta)) := \frac{1}{3}\int_\omega a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta\,\partial_{\alpha\beta}\eta \,\mathrm{d}\omega.
$$

Let $\chi \in H^2(\omega)$ be the unique solution of the variational equations $\int_\omega \Delta\chi\,\Delta\theta\,\mathrm{d}\omega = 0$ for all $\theta \in H_0^2(\omega)$, that also satisfies $\chi = \phi_0$ and $\partial_\nu \chi = \phi_1$ on γ. Finally, let $F \in V(\omega)$ denote the unique solution of the variational equations $((F, \eta)) = \int_\omega f\eta\,\mathrm{d}\omega$ for all $\eta \in V(\omega)$.

Then *finding a weak solution* (ξ, ϕ) *of the generalized von Kármán equations amounts to finding* $\xi \in V(\omega)$ *that satisfies the operator equation*:

$$\widetilde{C}(\xi) + \xi - \widetilde{B}(\chi, \xi) - F = 0 \quad \text{in } V(\omega),$$

where the *'cubic' mapping* $\widetilde{C} : V(\omega) \to V(\omega)$ is defined by $\widetilde{C}(\eta) := \widetilde{B}(B(\eta, \eta), \eta)$, for all $\eta \in V(\omega)$, the unknown $\phi \in H^2(\omega)$ being then given by $\phi = \chi - B(\xi, \xi)$. Naturally, finding the solution ξ of the above operator equation is equivalent to solving the following *variational problem*:

$$\xi \in V(\omega) \quad \text{and} \quad \big(\big(\widetilde{C}(\xi) + \xi - \widetilde{B}(\chi, \xi) - F, \eta\big)\big) = 0 \quad \text{for all } \eta \in V(\omega). \tag{P}$$

2. The discrete problem

We henceforth assume that the boundary of ω is a polygon, so that $\bar{\omega}$ can be exactly covered by a regular family of triangulations. Let $W_h \subset H^2(\omega)$, $V_h \subset V(\omega)$, and $V_{0h} \subset H_0^2(\omega)$ be standard conforming finite element spaces associated with such a family, i.e., that satisfy the minimal conditions of Theorem 6.1-7 in [1]. For each $h > 0$, the discrete problem is then defined through the following stages, which simply mimic those that lead to the operator equation satisfied by $\xi \in V(\omega)$:

Let the bilinear mapping $B_h : H^2(\omega) \times H^2(\omega) \to V_{0h}$ be defined as follows: Given $(\xi, \eta) \in H^2(\omega) \times H^2(\omega)$, the function $B_h(\xi, \eta) \in V_{0h}$ is the unique solution of the variational equations:

$$\int_\omega \Delta B_h(\xi, \eta)\Delta\theta_h\,\mathrm{d}\omega = \int_\omega [\xi, \eta]\theta_h\,\mathrm{d}\omega \quad \text{for all } \theta_h \in V_{0h}.$$

Define another bilinear mapping $\widetilde{B}_h : H^2(\omega) \times H^2(\omega) \to V_h$ as follows: Given $(\phi, \xi) \in H^2(\omega) \times H^2(\omega)$, the function $\widetilde{B}_h(\phi, \xi) \in V_h$ is the unique solution of the variational equations

$$\big(\big(\widetilde{B}_h(\phi, \xi), \eta_h\big)\big) = \int_\omega [\phi, \xi]\eta_h\,\mathrm{d}\omega \quad \text{for all } \eta_h \in V_h.$$

Let $\chi_h \in W_h$ be a standard finite element approximation of $\chi \in H^2(\omega)$, which therefore satisfies $\|\chi_h - \chi\|_{H^2(\omega)} \to 0$ as $h \to 0$. Finally, let $F_h \in V_h$ be the unique solution of the variational equations $((F_h, \eta_h)) = \int_\omega f\eta_h\,\mathrm{d}\omega$ for all $\eta_h \in V_h(\omega)$.

Then the *discrete problem* consists in finding $(\xi_h, \phi_h) \in V_h \times W_h$ in two stages: *First*, $\xi_h \in V_h$ is found by solving the *discrete operator equation*:

$$\widetilde{C}_h(\xi_h) + \xi_h - \widetilde{B}_h(\chi_h, \xi_h) - F_h = 0 \quad \text{in } V_h,$$

where the *discrete 'cubic' mapping* $\widetilde{C}_h : V_h \to V_h$ is defined by $\widetilde{C}_h(\eta_h) := \widetilde{B}_h(B_h(\eta_h, \eta_h), \eta_h)$ for all $\eta_h \in V_h$. Finding ξ_h is clearly equivalent to solving the following *discrete variational problem*:

$$\xi_h \in V_h \text{ and } \big(\big(\widetilde{C}_h(\xi_h) + \xi_h - \widetilde{B}_h(\chi_h, \xi_h) - F_h, \eta_h\big)\big) = 0 \quad \text{for all } \eta_h \in V_h, \tag{P$_h$}$$

which will be shown to have at least one solution in Theorem 3.1. *Second*, $\phi_h \in W_h$ is given by $\phi_h := \chi_h - B_h(\xi_h, \xi_h)$.

3. Convergence

The following theorem (whose proof is only briefly sketched here; see [5] for a complete proof) establishes the *convergence* of the finite element method described in Section 2. Interestingly, the same theorem automatically provides in addition the *existence* of a solution to the continuous problem (which otherwise can be established by a direct proof; see [6]). Strong and weak convergences are denoted $\to$ and $\rightharpoonup$ respectively. All convergences are meant to hold as h approaches zero. The notation $\|\cdot\|$ designates the norm associated with the inner-product $((\cdot, \cdot))$.

Theorem 3.1. *Assume that the norm $\|(\phi_0,\phi_1)\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ is small enough. Then there exists a constant M such that, for each $h>0$, the discrete variational problem (P_h) has at least one solution $\xi_h\in V_h$ that satisfies $\|\xi_h\|\leqslant M$. Let $(\xi_h)_{h>0}$ be any subsequence that weakly converges in $H^2(\omega)$ and let $\xi\in V(\omega)$ denote its limit. Then ξ is a solution of the variational equations (P), and*

$$\xi_h\to\xi\quad\text{in } H^2(\omega).$$

Sketch of proof. The proof consists in successively establishing the following properties:

(i) *The discrete cubic mapping $\widetilde{C}_h$ satisfies $(\!(\widetilde{C}_h(\eta_h),\eta_h)\!)\geqslant 0$ for all $\eta_h\in V_h$.* This property crucially hinges on the fact that the trilinear form

$$(\xi,\eta,\chi)\in H^2(\omega)\times H^2(\omega)\times H^2(\omega)\to\int_\omega[\xi,\eta]\chi\,\mathrm{d}\omega$$

becomes *symmetric* if at least one of its arguments is in $H_0^2(\omega)$ (for a proof, see, e.g., Theorem 5.8-2 in [3]).

(ii) *Let $\chi_h,\xi_h,\eta_h\in W_h$ be such that $\chi_h\to\chi$, $\xi_h\rightharpoonup\xi$, $\eta_h\rightharpoonup\eta$ in $H^2(\omega)$. Then $(\!(\widetilde{B}_h(\chi_h,\xi_h),\eta_h)\!)\to(\!(\widetilde{B}(\chi,\xi),\eta)\!)$.* This convergence is a consequence of the definitions of the bilinear operators $\widetilde{B}_h$ and $\widetilde{B}$ and of the compact inclusion of $H^2(\omega)$ into $\mathcal{C}^0(\bar{\omega})$.

(iii) *Let $\xi_h\in W_h$ be such that $\xi_h\rightharpoonup\xi$ in $H^2(\omega)$. Then $B_h(\xi_h,\xi_h)\to B(\xi,\xi)$ in $H_0^2(\omega)$.* This convergence relies in particular on the symmetry of the trilinear form considered in (i) and again on the compact inclusion of $H^2(\omega)$ into $\mathcal{C}^0(\bar{\omega})$.

(iv) *If the norm $\|(\phi_0,\phi_1)\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ is small enough, there exists a constant M independent of h such that problem (P_h) has at least one solution ξ_h that satisfies $\|\xi_h\|\leqslant M$.* This part of the proof is inspired by a crucial *compactness method of J.L. Lions* (see [10, Chapter 1, Lemma 4.3]). Let w_i^h, $1\leqslant i\leqslant d(h)$, be a basis of V_h that is orthonormal with respect to the inner product $(\!(.\,,.)\!)$ and let the mapping $\boldsymbol{G}^h=(G_i^h):\mathbb{R}^{d(h)}\to\mathbb{R}^{d(h)}$ be defined for each $\mathbf{X}=(X_i)\in\mathbb{R}^{d(h)}$ by

$$G_i^h(\mathbf{X}):=\big(\!\big(\widetilde{C}_h\big(\eta_h(\mathbf{X})\big)+\eta_h(\mathbf{X})-\widetilde{B}_h\big(\chi_h,\eta_h(\mathbf{X})\big)-F_h,w_i^h\big)\!\big),\quad 1\leqslant i\leqslant d(h),$$

where $\eta_h(\mathbf{X}):=\sum_{i=1}^{d(h)}X_iw_i^h$. Then after using (i) and performing some computations, one reaches the conclusion that there exist constants $c_1>0$ and $c_2>0$ independent of *h* such that ($\mathbf{X}\cdot\mathbf{Y}$ and $|\mathbf{X}|$ designate the Euclidean inner product of $\mathbf{X},\mathbf{Y}\in\mathbb{R}^{d(h)}$ and the Euclidean norm of $\mathbf{X}\in\mathbb{R}^{d(h)}$):

$$\boldsymbol{G}^h\big(\mathbf{X}\big)\cdot\mathbf{X}\geqslant\big(1-c_1\|\chi\|_{H^2(\omega)}\big)|\mathbf{X}|^2-c_2\|f\|_{L^2(\omega)}|\mathbf{X}|.$$

If $\|(\phi_0,\phi_1)\|_{H^{3/2}(\gamma)\times H^{1/2}(\gamma)}$ is small enough (so as to guarantee that $\|\chi\|_{H^2(\omega)}<c_1^{-1}$), there exists $M>0$ such that $\boldsymbol{G}^h(\mathbf{X})\cdot\mathbf{X}\geqslant 0$ for all $\mathbf{X}\in\mathbb{R}^{d(h)}$ that satisfy $|\mathbf{X}|=M$. The conclusion therefore follows from a well-known corollary to the *Brouwer fixed point theorem* (see, e.g., Lions [10, Chapter 1, Lemma 4.3]) and on the observation that $|\mathbf{X}|=\|\sum_{i=1}^{d(h)}X_iw_i^h\|$.

(v) *Let $(\xi_h)_{h>0}$ be any subsequence of the sequence found in* (iv) *that satisfies $\xi_h\rightharpoonup\xi$ in $H^2(\omega)$. Then ξ is a solution of the variational problem (P).* Given any $\eta\in V(\omega)$, let $\eta_h\in V_h$ be such that $\eta_h\to\eta$ *in $H^2(\omega)$*. Hence, for any $h>0$,

$$\big(\!\big(\widetilde{C}_h(\xi_h)+\xi_h-\widetilde{B}_h(\chi_h,\xi_h)-F_h,\eta_h\big)\!\big)=0.$$

First, it is clear that $(\!(\xi_h-F_h,\eta_h)\!)\to(\!(\xi-F,\eta)\!)$. Next, $(\!(\widetilde{B}_h(\chi_h,\xi_h),\eta_h)\!)\to(\!(B(\chi,\xi),\eta)\!)$ by (ii). Finally, part (ii) again shows that

$$\big(\!\big(\widetilde{C}_h(\xi_h),\eta_h\big)\!\big)=\big(\!\big(\widetilde{B}_h\big(B_h(\xi_h,\xi_h),\xi_h\big),\eta_h\big)\!\big)\to\big(\!\big(\widetilde{B}(B(\xi,\xi),\xi),\eta\big)\!\big)=\big(\!\big(\widetilde{C}(\xi),\eta\big)\!\big),$$

since $B_h(\xi_h,\xi_h)\to B(\xi,\xi)$ in $H_0^2(\omega)$ by (iii).

(vi) *The subsequence $(\xi_h)_{h>0}$ considered in* (v) *satisfies $\xi_h\to\xi$ in $H^2(\omega)$.* To see this, let $\eta_h=\xi_h$ in the variational equations of (P_h). Then $(\!(\widetilde{C}_h(\xi_h),\xi_h)\!)\to(\!(\widetilde{C}(\xi),\xi)\!)$ and $(\!(\widetilde{B}_h(\chi_h,\xi_h),\xi_h)\!)\to(\!(\widetilde{B}(\chi,\xi),\xi)\!)$ by (ii). It is then seen that this implies $\|\xi_h\|^2\to\|\xi\|$. □

4. Concluding remarks

(a) The functions $\phi_h = \chi_h - B_h(\xi_h, \xi_h)$ also converge strongly in $H^2(\omega)$ to the function ϕ, since $B_h(\xi_h, \xi_h) \to B(\xi, \xi)$ by part (iii) of the above proof.

(b) Since numerically finding the functions $\xi_h \in V_h$ amounts to finding a Brouwer fixed point, a continuation method of the form proposed by Kellogg, Li and Yorke [8] can be used for this purpose (see [5] for more details).

(c) A characteristic of the 'cubic' operator equation satisfied by ξ is the *loss of strict positivity* incurred by its cubic part, in the sense that $((\widetilde{C}(\eta), \eta)) \geqslant 0$ for all $\eta \in V(\omega)$, but $((\widetilde{C}(\eta), \eta))$ may be equal to zero for some non-zero $\eta \in V(\omega)$ (by contrast, the cubic part associated with the 'classical' von Kármán equations is strictly positive; cf. [3, Theorem 5.8-2]). This property precludes for instance the usage of a finite element method of the type proposed by Kesavan [9].

(d) Another feature of this problem is the *lack of symmetry of the bilinear form* $(\xi, \eta) \in V(\omega) \times V(\omega) \to ((\widetilde{B}(\chi, \xi), \eta))$ found in the same operator equation. This property prevents the usage of an associated functional (so as to reduce problem (P) to finding its minimizers, as in the case of the 'classical' von Kármán equations; cf. [3, Theorem 5.8-3]).

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References

[1] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[2] P.G. Ciarlet, A justification of the von Kármán equations, Arch. Rational Mech. Anal. 73 (1980) 349–389.
[3] P.G. Ciarlet, Mathematical Elasticity, vol. II: Theory of Plates, North-Holland, Amsterdam, 1997.
[4] P.G. Ciarlet, L. Gratie, Generalized von Kármán equations, J. Math. Pures Appl. 80 (2001) 263–279.
[5] P.G. Ciarlet, L. Gratie, S. Kesavan, On the generalized von Kármán equations and their approximation, in preparation.
[6] P.G. Ciarlet, L. Gratie, N. Sabu, An existence theorem for generalized von Kármán equations, J. Elasticity 62 (2001) 239–248.
[7] P.G. Ciarlet, P. Rabier, Les équations de von Kármán, Lecture Notes in Math., vol. 826, Springer-Verlag, Berlin, 1980.
[8] R.B. Kellogg, T.Y. Li, J. Yorke, A method of continuation for calculating a Brouwer fixed point, in: S. Karamardian (Ed.), Fixed Points: Algorithms and Applications, Academic Press, New York, 1977, pp. 133–147.
[9] S. Kesavan, La méthode de Kikuchi appliquée aux équations de von Kármán, Numer. Math. 32 (1979) 209–232.
[10] J.L. Lions, Quelques méthodes de résolution de problèmes aux limites non linéaires, Dunod, Paris, 1969.