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Partial Differential Equations/Mathematical Physics

The Camassa–Holm equation on the half-line

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Abstract

We study the initial-boundary-value problem for the Camassa–Holm equation on the half-line by associating to it a matrix Riemann–Hilbert problem in the complex k-plane; the jump matrix is determined in terms of the spectral functions corresponding to the initial and boundary values. We prove that if the boundary values $u(0,t)$ are $\geqslant 0$ for all t then the corresponding initial-boundary-value problem has a unique solution, which can be expressed in terms of the solution of the associated RH problem. In the case $u(0,t)<0$, the compatibility of the initial and boundary data is explicitly expressed in terms of an algebraic relation to be satisfied by the spectral functions. ***To cite this article: A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

L'équation de Camassa–Holm sur la demi-droite. Nous étudions un problème aux limites pour l'équation de Camassa–Holm sur la demi-droite en exprimant la solution en termes de la solution d'un problème de Riemann–Hilbert matriciel dans le plan complexe du paramètre spectral k. La matrice de saut de ce problème de Riemann–Hilbert est déterminée par les fonctions spectrales qui correspondent aux valeurs initiales et aux valeurs au bord. Nous démontrons que si les valeurs au bord $u(0,t)$ sont $\geqslant 0$ pout tout t, alors le problème aux limites a une solution unique, qui s'exprime en termes de la solution du problème de Riemann–Hilbert associé. Lorsque les valeurs au bord $u(0,t)$ sont <0, les valeurs aux limites doivent vérifier une relation de compatibilité qui s'explicite par une relation algébrique que doivent satisfaire les fonctions spectrales associées. ***Pour citer cet article : A. Boutet de Monvel, D. Shepelsky, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

L'objet de cette Note est d'étudier le problème aux limites pour l'équation de Camassa–Holm (CH) [2]

$$u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

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sur la demi-droite $x \geqslant 0$. Cette équation admet une « paire de Lax » constituée des équations linéaires

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m+1)\psi, \qquad \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi \tag{2}$$

où $m = u - u_{xx}$. La méthode que nous utilisons repose sur l'analyse spectrale simultanée de ces deux équations. C'est une variante de la méthode par transformation de « scattering » inverse utilisée pour le problème de Cauchy sur la droite [3]. Nous obtenons les résultats suivants.

Théorème 0.1. *Soit $u_0(x)$, $x \geqslant 0$, $\mathcal{C}^\infty$ à décroissance rapide, telle que $u_0(x) - u_{0xx}(x) + 1 > 0$ pour tout $x \geqslant 0$. Soient $v_0(t)$, $v_1(t)$, et $v_2(t)$, $t \in [0,T]$ des fonctions $\mathcal{C}^\infty$ telles que $v_0(t) > 0$, $v_0(t) - v_2(t) + 1 > 0$ pour tout t, et $(\partial_x)^j u_0(0) = v_j(0)$, $0 \leqslant j \leqslant 2$. Le problème aux limites défini par* (1) *et par les conditions*

$$\begin{aligned} &u(x,0) = u_0(x), \quad x > 0, \\ &(\partial_x)^j u(0,t) = v_j(t), \quad 0 < t < T,\ j = 0,1,2, \end{aligned} \tag{3}$$

a une solution unique, $u(x,t)$, telle que $u(x,t) - u_{xx}(x,t) + 1 > 0$ pour tout $x \in [0,\infty)$ et tout $t \in [0,T]$.

Remarque 1. Ce Théorème 0.1 vaut encore lorsque $v_0(t) \equiv 0$, à condition de prendre dans (3) comme nouvelles conditions en $x = 0$ les conditions $u(0,t) = 0$ et $u_x(0,t) = v_1(t)$.

Théorème 0.2. *On garde les hypothèses du Théorème* 0.1 *sauf qu'on suppose $v_0(t) < 0$ pour tout $t \in [0,T]$. Le problème aux limites* (1), (3) *a alors une solution unique pourvu que les fonctions spectrales $a(k)$, $b(k)$, $\tilde{A}(k)$ et $\widetilde{B}(k)$ associées aux données initiales et au bord satisfassent la « relation globale »* (9) *ci-après.*

1. Introduction

The Camassa–Holm (CH) equation [2]

$$u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

is a model for the unidirectional propagation of waves in shallow water. It is integrable, at least formally, because it possesses a Lax pair representation: the CH equation is the compatibility condition of two linear equations

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m+1)\psi, \qquad \psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi, \tag{2}$$

where $m = u - u_{xx}$. This, together with the fact that the x-equation of the Lax pair can be transformed, by using the Liouville transformation, to the spectral problem for the one-dimensional Schrödinger equation, allowed developing the inverse scattering transform method to study the initial value problem for the CH equation for initial data $u_0(x) = u(x,0)$, $-\infty < x < \infty$, vanishing, as $|x| \to \infty$, see [3].

Here, we study the initial-boundary-value problem for (1) on the half-line $0 \leqslant x < +\infty$, where the data are the initial values $u_0(x) = u(x,0)$, $0 \leqslant x < \infty$ vanishing rapidly as $x \to +\infty$, and the boundary values of $u(x,t)$ and of its normal derivatives for $x = 0$, $0 \leqslant t \leqslant T$.

Our approach can be viewed as a generalization of the inverse scattering transform method to the case of initial-boundary-value problems [1]. It is based on the *simultaneous* spectral analysis of the two eigenvalues equations of the Lax pair (2) in the domain $0 \leqslant x < +\infty$, $0 \leqslant t < T$. On the boundary of this domain, the analysis leads to the spectral problems either for the x-equation of the Lax pair (for $t = 0$) or for the t-equation (for $x = 0$). The solution of the initial-boundary-value problem is expressed in terms of the solution of a matrix Riemann–Hilbert (RH) factorization problem in the complex plane of the spectral parameter, the data for which are determined in terms of the spectral functions associated with the initial and boundary values of the solution.

Theorem 1.1. *Let $u_0(x)$, $x \in [0,\infty)$ be a smooth function, rapidly vanishing as $x \to \infty$, and such that $u_0(x) - u_{0xx}(x) + 1 > 0$ for all $x \in [0,\infty)$. Let $v_0(t)$, $v_1(t)$, and $v_2(t)$, $t \in [0,T]$ be smooth functions such that $v_0(t) > 0$, $v_0(t) - v_2(t) + 1 > 0$ for all t, and $(\partial_x)^j u_0(0) = v_j(0)$, $j = 0,1,2$. Then the initial-boundary-value problem defined by* (1) *with boundary conditions*

$$u(x,0)=u_0(x),\quad x>0,$$
$$(\partial_x)^j u(0,t)=v_j(t),\quad 0<t<T,\ j=0,1,2, \tag{3}$$

has a unique solution, $u(x,t)$*, such that* $u(x,t)-u_{xx}(x,t)+1>0$ *for all* $x\in[0,\infty)$ *and* $t\in[0,T]$.

Remark 1. In the case $v_0(t)\equiv 0$ Theorem 1.1 remains true, if the boundary conditions for $x=0$ in (3) are replaced by $u(0,t)=0$, $u_x(0,t)=v_1(t)$.

Theorem 1.2. *Assume that in the boundary conditions in Theorem* 1.1, $v_0(t)<0$ *for all* $t\in[0,T]$. *Then the initial-boundary-value problem* (1), (3) *has a unique solution provided the spectral functions* $a(k)$, $b(k)$ *and* $\tilde A(k)$, $\widetilde B(k)$ *associated with the initial and boundary conditions satisfy the relation* (9) *below.*

2. Eigenfunctions and spectral functions

We rewrite the Lax pair in a 2-vector form. Let $\Phi:=\binom{\psi}{\psi_x}$; then (2) is equivalent to

$$\Phi_x=\begin{pmatrix}0 & 1\\ \lambda(m+1)+\frac14 & 0\end{pmatrix}\Phi,\qquad \Phi_t=\begin{pmatrix}\frac{u_x}{2} & \frac{1}{2\lambda}-u\\ \frac{1}{8\lambda}+\frac12+\frac{u}{4}-\lambda u(m+1) & -\frac{u_x}{2}\end{pmatrix}\Phi. \tag{4}$$

In order to control the behavior of solutions of (4) as $\lambda\to 0$ and as $\lambda\to\infty$, it is convenient to transform (4) in such a way that the principal term appears as a diagonal matrix, whereas the term of order λ^0 is an off-diagonal matrix. Let $k:=\sqrt{-\lambda-1/4}$. Setting

$$\Phi_0:=\frac12\begin{pmatrix}1 & -1/\mathrm{i}k\\ 1 & 1/\mathrm{i}k\end{pmatrix}\Phi\exp\left\{\left(\mathrm{i}kx+\frac{\mathrm{i}k}{2\lambda}t\right)\sigma_3\right\}$$

gives

$$\Phi_{0x}+\mathrm{i}k[\sigma_3,\Phi_0]=U_0\Phi_0,\qquad \Phi_{0t}+\frac{\mathrm{i}k}{2\lambda}[\sigma_3,\Phi_0]=V_0\Phi_0, \tag{5}$$

where

$$\sigma_3=\operatorname{diag}\{1,-1\},\quad [a,b]:=ab-ba,\quad \mathrm{e}^{\hat\sigma_3}a:=\mathrm{e}^{\sigma_3}a\,\mathrm{e}^{-\sigma_3},$$
$$U_0(x,t,k)=\frac{\lambda}{2\mathrm{i}k}m(x,t)\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix},\quad V_0(x,t,k)=\frac{u_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\mp\frac{u}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}+\mathrm{O}(\lambda)\quad \text{as } k\to\pm\frac{\mathrm{i}}{2}.$$

Assume that there exists a smooth real-valued function $u(x,t)$ satisfying (1) in $\{0<x<\infty, 0<t<T\}$, such that $m(x,t)+1>0$ for all (x,t). Following (5), we define 2×2-matrix-valued eigenfunctions $\Phi_{0j}(x,t,k)$, $j=1,2$, as the solutions of the integral equations

$$\Phi_{01}(x,t,k)=I+\int_0^x \mathrm{e}^{-\mathrm{i}k(x-y)\hat\sigma_3}(U_0\Phi_{01})(y,t,k)\,\mathrm{d}y-\mathrm{e}^{-\mathrm{i}kx\hat\sigma_3}\int_t^T \mathrm{e}^{-\frac{\mathrm{i}k}{2\lambda}(t-\tau)\hat\sigma_3}(V_0\Phi_{01})(0,\tau,k)\,\mathrm{d}\tau,$$
$$\Phi_{02}(x,t,k)=I+\int_0^x \mathrm{e}^{-\mathrm{i}k(x-y)\hat\sigma_3}(U_0\Phi_{02})(y,t,k)\,\mathrm{d}y+\mathrm{e}^{-\mathrm{i}kx\hat\sigma_3}\int_0^t \mathrm{e}^{-\frac{\mathrm{i}k}{2\lambda}(t-\tau)\hat\sigma_3}(V_0\Phi_{02})(0,\tau,k)\,\mathrm{d}\tau.$$

The eigenfunctions $\Phi_{0j}(x,t,k)$ have well-controlled behavior near $k=\pm\mathrm{i}/2$ ($\lambda=0$). Defining

$$\Phi_\infty(x,t,k):=\frac12\begin{pmatrix}1 & -1/\mathrm{i}k\\ 1 & 1/\mathrm{i}k\end{pmatrix}\begin{pmatrix}(m+1)^{1/4} & 0\\ 0 & (m+1)^{-1/4}\end{pmatrix}\Phi\,\mathrm{e}^{\mathrm{i}kp(x,t)\sigma_3}$$

with

$$p(x,t)=\int_0^x\sqrt{m(\xi,t)+1}\,\mathrm{d}\xi-\int_0^t u(0,\zeta)\sqrt{m(0,\zeta)+1}\,\mathrm{d}\zeta$$

gives a form of the Lax pair convenient for the introduction of eigenfunctions well-controlled for large λ:

$$\Phi_{\infty x} + \mathrm{i}k\sqrt{m+1}\,[\sigma_3, \Phi_\infty] = U_\infty \Phi_\infty, \qquad \Phi_{\infty t} - \mathrm{i}ku\sqrt{m+1}\,[\sigma_3, \Phi_\infty] = V_\infty \Phi_\infty, \tag{6}$$

where $U_\infty(x,t,k) = \frac{1}{4}\frac{m_x}{m+1}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) + \mathrm{O}(1/k)$ and $V_\infty(x,t,k) = -\frac{u}{4}\frac{m_x}{m+1}\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) + \mathrm{O}(1/k)$ as $k \to \infty$. The eigenfunctions $\Phi_{\infty j}(x,t,k)$, $j = 1,2,3$ are defined as the solutions of the integral equations

$$\begin{aligned}
\Phi_{\infty 1}(x,t,k) = I &+ \int_0^x \mathrm{e}^{-\mathrm{i}k\int_y^x \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat\sigma_3}(U_\infty \Phi_{\infty 1})(y,t,k)\,\mathrm{d}y \\
&- \mathrm{e}^{-\mathrm{i}k\int_0^x \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat\sigma_3} \int_t^T \mathrm{e}^{-\mathrm{i}k\int_t^\tau u(0,\zeta)\sqrt{m(0,\zeta)+1}\,\mathrm{d}\zeta\,\hat\sigma_3}(V_\infty \Phi_{\infty 1})(0,\tau,k)\,\mathrm{d}\tau, \\
\Phi_{\infty 2}(x,t,k) = I &+ \int_0^x \mathrm{e}^{-\mathrm{i}k\int_y^x \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat\sigma_3}(U_\infty \Phi_{\infty 2})(y,t,k)\,\mathrm{d}y \\
&+ \mathrm{e}^{-\mathrm{i}k\int_0^x \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat\sigma_3} \int_0^t \mathrm{e}^{\mathrm{i}k\int_\tau^t u(0,\zeta)\sqrt{m(0,\zeta)+1}\,\mathrm{d}\zeta\,\hat\sigma_3}(V_\infty \Phi_{\infty 2})(0,\tau,k)\,\mathrm{d}\tau, \\
\Phi_{\infty 3}(x,t,k) = I &- \int_x^\infty \mathrm{e}^{\mathrm{i}k\int_x^y \sqrt{m(\xi,t)+1}\,\mathrm{d}\xi\,\hat\sigma_3}(U_\infty \Phi_{\infty 3})(y,t,k)\,\mathrm{d}y.
\end{aligned}$$

Now the spectral functions are defined as matrix-valued functions of k relating the eigenfunctions introduced above:

$$\begin{aligned}
\Phi_{01}(x,t,k) &= \Phi_{02}(x,t,k)\,\mathrm{e}^{-\mathrm{i}(kx+\frac{k}{2\lambda}t)\hat\sigma_3} S(k), \\
\Phi_{\infty 1}(x,t,k) &= \Phi_{\infty 2}(x,t,k)\,\mathrm{e}^{-\mathrm{i}kp(x,t)\hat\sigma_3} \widetilde{S}(k), \\
\Phi_{\infty 3}(x,t,k) &= \Phi_{\infty 2}(x,t,k)\,\mathrm{e}^{-\mathrm{i}kp(x,t)\hat\sigma_3} s(k),
\end{aligned} \tag{7}$$

where

$$S(k) \equiv \begin{pmatrix} \overline{A(\bar k)} & B(k) \\ \overline{B(\bar k)} & A(k) \end{pmatrix} = \Phi_{01}(0,0,k), \qquad s(k) \equiv \begin{pmatrix} \overline{a(\bar k)} & b(k) \\ \overline{b(\bar k)} & a(k) \end{pmatrix} = \Phi_{\infty 3}(0,0,k),$$

$$\widetilde{S}(k) \equiv \begin{pmatrix} \overline{\tilde A(\bar k)} & \widetilde{B}(k) \\ \overline{\widetilde{B}(\bar k)} & \tilde A(k) \end{pmatrix} = \Phi_{\infty 1}(0,0,k).$$

The boundary values of u and the spectral functions are related by the direct and inverse spectral maps $\{u_0(x)\} \leftrightarrow \{a(k), b(k)\}$ and $\{v_0(t), v_1(t), v_2(t)\} \leftrightarrow \{A(k), B(k)\}$ (or $\{v_0(t), v_1(t), v_2(t)\} \leftrightarrow \{\tilde A(k), \widetilde{B}(k)\}$), the inverse maps being described in terms of the associated Riemann–Hilbert problems [1]. Evaluating $\Phi_{\infty 3}(x,t,k)$ in (7) at $x=0$, $t=T$ we obtain

$$\Phi_{\infty 3}(0,T,k) = \mathrm{e}^{-\mathrm{i}kp(0,T)\sigma_3} \widetilde{S}^{-1}(k) s(k)\,\mathrm{e}^{\mathrm{i}kp(0,T)\sigma_3}, \tag{8}$$

where $p(0,T) = \int_0^T v_0(\zeta)\sqrt{v_0(\zeta) - v_2(\zeta) + 1}\,\mathrm{d}\zeta$. In particular, the analytic and asymptotic properties of $\Phi_{\infty 3}$ imply that the $(1,2)$-entry of the r.h.s. of (8) is analytic in the domain $\operatorname{Im} k > 0$, $|k| > R$ and

$$\bigl(\tilde A(k) b(k) - \widetilde{B}(k) a(k)\bigr)\,\mathrm{e}^{2\mathrm{i}k\int_0^T v_0(\zeta)\sqrt{v_0(\zeta)-v_2(\zeta)+1}\,\mathrm{d}\zeta} = \mathrm{O}(1/k), \quad |k| \to \infty,\ \operatorname{Im} k > 0. \tag{9}$$

3. The Riemann–Hilbert problem

Taking into account the analytic properties of the eigenfunctions, the scattering relations (7) can be rewritten in terms of a family of Riemann–Hilbert problems (parametrized by x and t), with a jump matrix determined by the spectral functions. Let $D_1 := \{k \mid |k - \frac{\mathrm{i}}{2}| \leqslant \varepsilon\}$ and $D_2 := \{k \mid |k + \frac{\mathrm{i}}{2}| \leqslant \varepsilon\}$, for sufficiently small $\varepsilon > 0$. Let Σ be the contour $\Sigma := \{\operatorname{Im} k = 0\} \cup \{|k| = 1/2\} \cup \{|k| = R\} \cup \partial D_1 \cup \partial D_2$, with sufficiently large R. *Assume* that $u(0,t) \leqslant 0$. Let

$$Q(x,t) := \frac{1}{2}\big(m(x,t)+1\big)^{1/4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}\big(m(x,t)+1\big)^{-1/4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Define a sectionally meromorphic, 2×2-matrix-valued function ($\Phi^{(j)}$ denotes the jth-column of a 2×2 matrix Φ)

$$M(x,t,k) = \begin{cases} \begin{pmatrix} \Phi_{01}^{(1)}(x,t,k) & \dfrac{\Phi_{02}^{(2)}(x,t,k)}{\overline{A(\bar{k})}} \end{pmatrix} & \text{if } k \in D_1,\ |k| < \dfrac{1}{2},\ \text{or } k \in D_2,\ |k| > \dfrac{1}{2}, \\ \begin{pmatrix} \dfrac{\Phi_{02}^{(1)}(x,t,k)}{A(k)} & \Phi_{01}^{(2)}(x,t,k) \end{pmatrix} & \text{if } k \in D_1,\ |k| > \dfrac{1}{2},\ \text{or } k \in D_2,\ |k| < \dfrac{1}{2}, \\ \Phi_{02}(x,t,k) & \text{if } |k| < R,\ k \notin D_1 \cup D_2, \\ Q^{-1}(x,t)\begin{pmatrix} \dfrac{\Phi_{\infty 2}^{(1)}(x,t,k)}{a(k)} & \Phi_{\infty 3}^{(2)}(x,t,k) \end{pmatrix} & \text{if } |k| > R,\ \operatorname{Im} k > 0, \\ Q^{-1}(x,t)\begin{pmatrix} \Phi_{\infty 3}^{(1)}(x,t,k) & \dfrac{\Phi_{\infty 2}^{(2)}(x,t,k)}{\overline{a(\bar{k})}} \end{pmatrix} & \text{if } |k| > R,\ \operatorname{Im} k < 0. \end{cases}$$

The limits $M_\pm(x,t,k)$ of $M(x,t,k')$ as k' approaches $k \in \Sigma$ from the adjacent sectors are related by

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \Sigma, \tag{10}$$

where the jump matrix $J(x,t,k)$ is defined as follows:

$$J(x,t,k) = \begin{cases} \mathrm{e}^{-\mathrm{i}kp(x,t)\sigma_3}\begin{pmatrix} 1 & -\dfrac{b(k)}{\overline{a(k)}} \\ \dfrac{\overline{b(k)}}{a(k)} & \dfrac{1}{|a(k)|^2} \end{pmatrix}\mathrm{e}^{\mathrm{i}kp(x,t)\sigma_3} & \text{if } \operatorname{Im} k = 0,\ |k| > R, \\ \mathrm{e}^{-\mathrm{i}kp(x,t)\sigma_3}\begin{pmatrix} a(k) & -b(k) \\ 0 & \dfrac{1}{a(k)} \end{pmatrix} Q(0,0)\mathrm{e}^{\mathrm{i}(kx+\frac{k}{2\lambda}t)\sigma_3} & \text{if } \operatorname{Im} k > 0,\ |k| = R, \\ \mathrm{e}^{-\mathrm{i}(kx+\frac{k}{2\lambda}t)\sigma_3}\begin{pmatrix} \dfrac{\overline{A(\bar{k})}}{B(\bar{k})} & 0 \\ & \dfrac{1}{\overline{A(\bar{k})}} \end{pmatrix}\mathrm{e}^{\mathrm{i}(kx+\frac{k}{2\lambda}t)\sigma_3} & \text{if } \left|k - \dfrac{\mathrm{i}}{2}\right| = \varepsilon,\ |k| < \dfrac{1}{2}, \\ \mathrm{e}^{-\mathrm{i}(kx+\frac{k}{2\lambda}t)\sigma_3}\begin{pmatrix} 1 & -\dfrac{B(k)}{\overline{A(\bar{k})}} \\ \dfrac{\overline{B(\bar{k})}}{A(k)} & \dfrac{1}{A(k)\overline{A(\bar{k})}} \end{pmatrix}\mathrm{e}^{\mathrm{i}(kx+\frac{k}{2\lambda}t)\sigma_3} & \text{if } \left|k - \dfrac{\mathrm{i}}{2}\right| < \varepsilon,\ |k| = \dfrac{1}{2}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\big(\overline{J(x,t,\bar{k})}\big)^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \operatorname{Im} k < 0, \\ I & \text{if } \operatorname{Im} k = 0,\ |k| < R, \\ I & \text{if } |k| = \dfrac{1}{2},\ k \notin D_1 \cup D_2. \end{cases} \tag{11}$$

The definitions of M and J for the case $u(0,t) > 0$ are similar.

4. Sketch of proofs

The jump relation (10) suggests the definition of the matrix-valued function $J^{(z)}(z,x,t,k)$ by (11), with $z - \int_0^t v_0(\zeta)\sqrt{v_0(\zeta) - v_2(\zeta) + 1}\,\mathrm{d}\zeta$ instead of $p(x,t)$. Let $M^{(z)}(z,x,t,k)$, $x, z \in [0,\infty)$, $t \in [0,T]$ be the solution of the following Riemann–Hilbert problem:

- $M^{(z)}(z,x,t,k)$ is piecewise holomorphic relative to the contour Σ;
- $M_-^{(z)}(z,x,t,k) = M_+^{(z)}(z,x,t,k)J^{(z)}(z,x,t,k)$ for $k \in \Sigma$;
- $M^{(z)}(z,x,t,k) = I + \mathrm{O}(1/k)$ as $k \to \infty$.

Let $\mu(z,x,t) = (M_{12}^{(z)}(z,x,t,\mathrm{i}/2) + M_{22}^{(z)}(z,x,t,\mathrm{i}/2))^2$. Define $z = z(x,t)$ as the solution of the differential equation

$$\frac{\partial}{\partial x} z(x,t) = \mu(z,x,t), \quad z(0,t) = 0.$$

Set $m(x,t) = (\mu(z(x,t),x,t))^2 - 1$. Finally, determine $u(x,t)$ by

$$u(x,t) = \frac{1}{2}\left\{ \int_0^x \mathrm{e}^{y-x} m(y,t)\,\mathrm{d}y + \int_x^\infty \mathrm{e}^{x-y} m(y,t)\,\mathrm{d}y \right\} + \mathrm{e}^{-x}\left\{ v_0(t) - \int_0^\infty \mathrm{e}^{-y} m(y,t)\,\mathrm{d}y \right\}.$$

The Riemann–Hilbert problem formulated above is uniquely solvable for all x and t. The proof that u solves the Camassa–Holm equation is based on the dressing method [4]. The proof that u satisfies the initial and boundary conditions is based on the fact that the RH problem for $M^{(z)}(0,0,t,k)$ (respectively, $M^{(z)}(z,x,0,k)$) is equivalent to the RH problem describing the inverse spectral map $\{A,B\} \to \{v_0,v_1,v_2\}$ (respectively, $\{a,b\} \to \{u_0\}$); in the case $u(0,t) < 0$, the proof of this equivalence requires (9).

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References

[1] A. Boutet de Monvel, D. Shepelsky, Initial boundary value problem for the mKdV equation on a finite interval, Ann. Inst. Fourier 54 (2004) 1477–1495.
[2] R. Camassa, D. Holm, An integrable shallow water equation with peaked solutions, Phys. Rev. Lett. 71 (1993) 1661–1664.
[3] A. Constantin, On the scattering problem for the Camassa–Holm equation, Proc. Roy. Soc. London Ser. A 457 (2001) 953–970.
[4] V.E. Zakharov, A.B. Shabat, A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, Funct. Anal. Appl. 8 (1974) 226–235.