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Mathematical Analysis

Matrix-theoretical derivations of some results of Borcea–Shapiro on hyperbolic polynomials

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Abstract

We use matrix analysis to give simple proofs of two theorems of Borcea–Shapiro which yield majorization relations between certain hyperbolic polynomials. We also prove a conjecture of Borcea involving majorization and the zeros of polynomials. ***To cite this article: R. Pereira, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

La dérivation matricielle de certaines résultats de Borcea–Shapiro sur les polynômes hyperboliques. On utilise l'analyse matricielle pour obtenir des démonstrations simples de deux résultats de Borcea–Shapiro sur la relation de majoration entre certains polynômes hyperboliques. On obtient aussi un résultat apparenté sur la majoration des zéros de polynômes complexes. ***Pour citer cet article : R. Pereira, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

We will use much the same terminology as in [4]. For any nth degree polynomial p, let $Z(p)$ be the unordered n-tuple consisting of the zeros of p with each zero occurring as many times as its multiplicity. We say that a polynomial p is hyperbolic if all of its zeros are real. Let $\mathcal{H}_n$ be the set of monic hyperbolic nth degree polynomials. For any real number α, let D_α be the differential operator $(1-\alpha\frac{\mathrm{d}}{\mathrm{d}x})\,\mathrm{e}^{\alpha\frac{\mathrm{d}}{\mathrm{d}x}}$ (hence $D_\alpha p(x) = p(x+\alpha) - \alpha p'(x+\alpha)$.) We note that D_α maps $\mathcal{H}_n$ into itself. (An elementary exposition of this and related results can be found in [2]). We also remind the reader of the definition of the majorization relation between n-tuples of real numbers.

Definition 1. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers and let $(a_1^*, a_2^*, \ldots, a_n^*)$ and $(b_1^*, b_2^*, \ldots, b_n^*)$ be $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$, respectively, arranged in descending order. We then say $(a_1, a_2, \ldots, a_n)$ is majorized by $(b_1, b_2, \ldots, b_n)$ (and we write $(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$) if (i) $\sum_{i=1}^k a_i^* \leqslant \sum_{i=1}^k b_i^*$ for all k; $1 \leqslant k \leqslant n-1$ and (ii) $\sum_{i=1}^n a_i^* = \sum_{i=1}^n b_i^*$.

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[6] is the standard reference for majorization. The following well-known result on majorization will be useful. Here S_n denotes the group of permutations on n elements.

Proposition 2. *Let I be any interval in $\mathbb{R}$ and let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two n-tuples of real numbers in I. Then the following are equivalent*:

(1) $(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$.
(2) $(a_1, a_2, \ldots, a_n)$ *is in the convex hull of* $\{(b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(n)})\}_{\sigma \in S_n}$.
(3) $\sum_{i=1}^n \phi(a_i) \leqslant \sum_{i=1}^n \phi(b_i)$ *for all convex functions* $\phi : I \to \mathbb{R}$.

For $p, q \in \mathcal{H}_n$, we write $p \preccurlyeq q$ if $Z(p) \prec Z(q)$.

We now review some key concepts from [7]. Throughout this paper, let $L(\mathbb{C}^n)$ denote the set of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^n$. If $P \in L(\mathbb{C}^n)$ is a projection, $P\mathbb{C}^n$ denotes the range of P. If $A \in L(\mathbb{C}^n)$, let $\det(A)$ denote the determinant of A and $\operatorname{tr}(A)$ denote the trace of A.

Definition 3. Let $A \in L(\mathbb{C}^n)$, P be a projection from $\mathbb{C}^n$ onto a subspace of $\mathbb{C}^n$ having co-dimension one, and $B = PAP|_{P\mathbb{C}^n}$. Then we shall say that P is a differentiator of A if $\det(B - xI) = -\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\det(A - xI)$ for all $x \in \mathbb{C}$.

Definition 4. Let $A \in L(\mathbb{C}^n)$ and $v \in \mathbb{C}^n$. Then we say that v is a trace vector of A if $v^* A^i v = \frac{1}{n}\operatorname{tr}(A^i)$ for $0 \leqslant i < n$.

It is implicit in the definition that every trace vector has unit length. It was shown in [7] that any $A \in L(\mathbb{C}^n)$ has a trace vector. The following result (Theorem 2.5 from [7]) shows that differentiators and trace vectors are related.

Proposition 5. *Let $A \in L(\mathbb{C}^n)$, $v \in \mathbb{C}^n$ and P be the projection onto the orthogonal complement of the span of v. Then P is a differentiator of A if and only if v is a trace vector of A.*

We can now prove our main theorem.

Theorem 6. *Let $A \in L(\mathbb{C}^n)$, $p(x)$ be the characteristic polynomial of A and v be a trace vector of A. Let $\alpha \in \mathbb{C}$; then the characteristic polynomial of $A + n\alpha vv^*$ is $(1 - \alpha\frac{\mathrm{d}}{\mathrm{d}x})p(x)$.*

Proof. Let P be the projection onto the orthogonal complement of the span of v and let $B = PAP|_{P\mathbb{C}^n}$. Using the multilinearity property of the determinant with respect to any basis of $\mathbb{C}^n$ containing v, we find that $\det(A + n\alpha vv^* - xI) = \det(A - xI) + n\alpha \det(B - xI) = p(x) - \alpha p'(x)$.

Let w denote the n-vector all of whose entries are $\frac{1}{\sqrt{n}}$; it can easily be verified that w is a trace vector for any n by n diagonal matrix. Let $J = nww^*$ denote the n by n matrix all of whose entries are one. It follows from Theorem 6 that if A is any n by n diagonal matrix with characteristic polynomial p, then the characteristic polynomial of $A + \alpha J$ is $p(x) - \alpha p'(x)$ and the characteristic polynomial of $A_\alpha = A + \alpha(J - I)$ is $D_\alpha p$. We are now ready to prove Theorem 1.4. from [4]. □

Theorem 7. *If $p \in \mathcal{H}_n$ then $p \preccurlyeq D_\alpha p$ for any $\alpha \in \mathbb{R}$.*

Proof. Let A be an n by n diagonal matrix whose characteristic polynomial is p. Then $A_\alpha = A + \alpha(J - I)$ is a Hermitian matrix whose diagonal elements are the zeros of p and whose eigenvalues are the zeros of $D_\alpha p$. The theorem now follows from the result of Schur that the diagonal elements of a Hermitian matrix are majorized by its eigenvalues. □

Theorem 1.2. from [4] can be proven by using the same technique.

Theorem 8. *Let $p, q \in \mathcal{H}_n$ be such that $p \preccurlyeq q$. Then $(p + \alpha p') \preccurlyeq (q + \alpha q')$ for any $\alpha \in \mathbb{R}$.*

Proof. Let $r_1, r_2, \ldots, r_n$ and $s_1, s_2, \ldots, s_n$ be the zeros of p and q respectively listed in descending order. Let $A_p = \operatorname{diag}(r_1, r_2, \ldots, r_n)$. For every permutation $\sigma \in S_n$, let $A_\sigma = \operatorname{diag}(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})$. Since $p \preccurlyeq q$, there exist $\lambda_\sigma \geqslant 0$ for all $\sigma \in S_n$ with $\sum_{\sigma \in S_n} \lambda_\sigma = 1$ such that $A_p = \sum_{\sigma \in S_n} \lambda_\sigma A_\sigma$. Therefore $A_p - \alpha J = \sum_{\sigma \in S_n} \lambda_\sigma (A_\sigma - \alpha J)$. The required result now follows from the following theorem of Ky Fan [5]: $\square$

Proposition 9. *Let $\{A_i\}_{i=1}^k$ be a set of n by n Hermitian matrices and let $A = \sum_{i=1}^k A_i$. Let $\lambda(A)$ and $\lambda(A_i)$ be n-tuples whose zeros are the eigenvalues of A and A_i listed in descending order. Then $\lambda(A) \prec \sum_{i=1}^k \lambda(A_i)$.*

We note that the polynomials in the previous two theorems must be hyperbolic and α must be restricted to the real line for the two previous proofs since otherwise some of the matrices involved would cease being Hermitian and the classical majorization results no longer apply. In [4], it is conjectured (Conjecture 3.3) that for any monic polynomial p with complex zeros $\Re Z(p) \prec \Re Z(D_\alpha p)$ for any $\alpha \in \mathbb{R}$. (Here $\Re z = \frac{z+\bar{z}}{2}$ and $\Im z = \frac{z-\bar{z}}{2\mathrm{i}}$ denote the real and imaginary parts of z.) This conjecture is false even if restricted to polynomials p having only real coefficients. One family of counterexamples are the finite sections of the exponential series: $p_n = (n!) \sum_{i=1}^n \frac{x^i}{i!}$ when $n \geqslant 3$. It can easily be verified that $D_1 p_n = (x+1)^n$ and since p_n does not have all of its zeros on the line $\Re z = -1$ when $n \geqslant 3$, the conjecture is false.

In [3], one finds a weaker conjecture that either $\Re Z(p) \prec \Re Z(D_\alpha p)$ or $\Im Z(D_\alpha p) \prec \Im Z(p)$ for all complex polynomials p and all $\alpha \in \mathbb{R}$. We prove this conjecture by showing that the second relation always holds.

Theorem 10. *Let p be any nonconstant polynomial with complex coefficients, then $\Im Z(D_\alpha p) \prec \Im Z(p)$ for all $\alpha \in \mathbb{R}$.*

We can of course replace the $D_\alpha p$ in the statement of the theorem by $p - \alpha p'$. This result can be viewed as a generalization of the fact that D_α maps $\mathcal{H}_n$ to itself for all real α.

Proof. Let $r_1, r_2, \ldots, r_n$ denote the zeros of p. Let $A = \operatorname{diag}(r_1, r_2, \ldots, r_n) + \alpha(I - J)$ and use the following result of Amir-Moéz and Horn ([1], Theorem 2). Note that $\frac{A-A^*}{2\mathrm{i}} = \operatorname{diag}(\Im r_1, \Im r_2, \ldots, \Im r_n)$. $\square$

Lemma 11. *Let A be an n by n matrix, let $\Im\lambda(A)$ denote the unordered n-tuple consisting of the imaginary parts of the eigenvalues of A and let $\lambda(\frac{A-A^*}{2\mathrm{i}})$ denote the unordered n-tuple consisting of the eigenvalues of $\frac{A-A^*}{2\mathrm{i}}$. Then $\Im\lambda(A) \prec \lambda(\frac{A-A^*}{2\mathrm{i}})$.*

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