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Group Theory/Lie Algebras

Isotriviality of torsors over Laurent polynomial rings

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Abstract

We establish the isotriviality of torsors for Laurent polynomial rings in characteristic 0, and indicate an application to extended affine Lie algebras. ***To cite this article: P. Gille, A. Pianzola, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Isotrivialité des torseurs sur les anneaux de polynômes de Laurent. Nous établissons l'isotrivialité des torseurs sur les anneaux de polynômes de Laurent en caractéristique nulle et indiquons une application aux algèbres de Lie affines étendues. ***Pour citer cet article : P. Gille, A. Pianzola, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soient k un corps de caractéristique nulle et R_n la k-algèbre $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ des polynômes de Laurent en n indéterminées $t_1, \ldots, t_n$.

Cette Note se propose d'étudier des questions d'isotrivialité pour les algèbres de Lie affines étendues. (Qui sont des analogues de nullité supérieure des algèbres de Kac–Moody affines [1].) De façon plus précise [4], on s'intéresse aux k-algèbres de Lie $\mathcal{L}$ satisfaisant

(1) $\mathcal{L}$ est une R_n-algèbre de Lie (pour un certain n) telle que

$$R_n \simeq C_k(\mathcal{L}) := \left\{ \chi \in \mathrm{End}_{k\text{-mod}}(\mathcal{L}) \mid \chi\left([x, y]\right) = \left[x, \chi(y)\right] \text{ pour tous } x, y \in \mathcal{L} \right\} ;$$

(2) Il existe une algèbre de Lie simple déployée $\mathfrak{g}$ de dimension finie (unique à isomorphisme près) et une extension fidèlement plate de présentation finie S/R_n telle que $\mathcal{L} \otimes_{R_n} S \simeq_S \mathfrak{g} \otimes_k S$ (isomorphisme de S-algèbres de Lie).

La théorie de la descente montre qu'une telle algèbre $\mathcal{L}$ définit un R_n-torseur sous le k-groupe algébrique linéaire $\mathbf{Aut}(\mathfrak{g})$ et donc une classe $[\mathcal{L}] \in H^1(R_n, \mathbf{Aut}(\mathfrak{g}))$, où $H^1(R_n, \mathbf{Aut}(\mathfrak{g}))$ désigne le H^1 non abélien pour la cohomologie *fppf* (ou étale). Le résultat principal de cette Note est le suivant.

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Théorème 3.1. *On suppose k algébriquement clos. Pour tout entier $d \geqslant 1$, on pose $R_{n,d} = k[t_1^{\pm 1/d}, t_2^{\pm 1/d}, \dots t_n^{\pm 1/d}]$, et on note $R_{n,\infty} = \varinjlim_d R_{n,d}$. Alors pour tout k-groupe algébrique linéaire $\mathbf{G}$, on a $H^1(R_{n,\infty}, \mathbf{G}) = 1$.*

Ceci permet de retrouver que $R_{n,\infty}/R_n$ est le revêtement universel de R_n (cf. Corollaire 3.2). Ainsi, notre résultat signifie que les R_n-torseurs sous $\mathbf{G}$ sont *isotriviaux*, c'est-à-dire trivialisés par un revêtement galoisien fini convenable de R_n selon la terminologie de [11] §2.3. Ces torseurs s'explicitent donc en termes de cocycles galoisiens, i.e. $H^1(\pi_1(R_n, 1), \mathbf{G}(R_{n,\infty})) \cong H^1(R_n, \mathbf{G})$. En particulier, les R_n-algèbres de Lie $\mathcal{L}$ décrites ci-dessus sont isotriviales, i.e. satisfont $\mathcal{L} \otimes_{R_n} R_{n,\infty} \simeq_{R_{n,\infty}} \mathfrak{g} \otimes_k R_{n,\infty}$ et peuvent être décrites en termes de cocycles galoisiens.

La preuve du théorème nécessite un résultat préliminaire, à savoir la trivialité de $H^1_{Zar}(R_n, \mathbf{G})$ correspondant aux torseurs qui sont localement trivaux pour la topologie de Zariski (cf. Proposition 4.6). Si $\mathbf{G}$ est le groupe linéaire, ce fait est une variante du théorème de Quillen–Suslin. En général, c'est une application du théorème de Raghunathan $H^1(k[t_1, \dots, t_n], \mathbf{G}) = 1$ [7]. Une fois ce fait établi, la démonstration du Théorème 3.1 passe par une récurrence sur n s'appuyant sur le cas fondamental $n = 1$.

1. Introduction

Throughout k will denote a field of characteristic 0, and R_n the k-algebra $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomial on the n variables $t_1, \dots, t_n$.

This Note is concerned with isotriviality questions that arise naturally in the study of Extended Affine Lie Algebras (which are, roughly speaking, higher nullity analogues of the affine Kac–Moody Lie algebras. See [1]). More precisely [4], one is naturally lead to consider Lie algebras $\mathcal{L}$ over k with the following two properties.

(1) The centroid $C_k(\mathcal{L})$ of $\mathcal{L}$ (see [5] Chapter X) is equipped with an isomorphism $C_k(\mathcal{L}) \simeq R_n$ for some n.
(2) There exists a finite dimensional split simple Lie algebra $\mathfrak{g}$ (unique up to isomorphism), and a faithfully flat and finitely presented ring extension S of $C_k(\mathcal{L})$ such that $\mathcal{L} \otimes_{C_k(\mathcal{L})} S \simeq_S \mathfrak{g} \otimes_k S$ (isomorphism of S-Lie algebras).

Thus $\mathcal{L}$, which is in general infinite dimensional over k, can be viewed as a 'algebraic' object over its centroid, yielding thereof an R_n-torsor under the linear algebraic group $\mathbf{Aut}(\mathfrak{g})$.[1]

For any positive d, let $R_{n,d} = k[t_1^{\pm 1/d}, t_2^{\pm 1/d}, \dots, t_n^{\pm 1/d}]$, and set $R_{n,\infty} = \varinjlim_d R_{n,d}$. At the fraction field level, we let $K_{n,d} = k(t_1^{\pm 1/d}, t_2^{\pm 1/d}, \dots, t_n^{\pm 1/d})$ and $K_{n,\infty} = \varinjlim_d K_{n,d}$. Denote by 1_n the closed subset of $\mathrm{Spec}(R_{n,\infty})$ whose only element is the projective limit of the points $1 \in \mathrm{Spec}(R_{n,d})$.

2. Torsors

Let S be scheme and $\mathbf{G}/S$ a group scheme. Recall that a *right S-torsor under* $\mathbf{G}$ (or simply a $\mathbf{G}$-*torsor)*, is a scheme X/S together with a right action $X \times_S \mathbf{G} \to X$ such that

(a) The morphism $X \to S$ is a covering morphism for the *fppf*-topology.
(b) The morphism $X \times_S \mathbf{G} \to X \times_S X$, $(x, g) \mapsto (x, xg)$ is an isomorphism.

The *trivial* $\mathbf{G}$-*torsor* is $\mathbf{G}$ acting on itself by right translation. Let X be a $\mathbf{G}$-torsor. By definition X is trivialized (i.e. becomes isomorphic to the trivial torsor) by an *fppf* covering of S (namely X/S). We say X is *locally trivial* (resp. *étale locally trivial*), if it admits a trivialization by a Zariski (resp. étale) open covering of S. If $\mathbf{G}/S$ is smooth, any $\mathbf{G}$-torsor is étale locally trivial (cf. [10], Exp. XXIV, as well as [11]). If $\mathbf{G}/S$ is affine and flat of finite type, $\mathbf{G}$-torsors (resp. locally trivial $\mathbf{G}$-torsors, resp. étale locally trivial $\mathbf{G}$-torsors) are classified by the pointed set of cohomology $H^1_{fppf}(S, \mathbf{G})$ (resp. $H^1_{Zar}(S, \mathbf{G})$, resp. $H^1_{ét}(S, \mathbf{G})$) defined by means of cocycles à la Čech. We denote henceforth $H^1_{ét}(S, \mathbf{G})$ simply by $H^1(S, \mathbf{G})$. If $S_0 \subset S$ is a closed subscheme, we define $H^1(S, S_0, \mathbf{G}) := \ker(H^1(S, \mathbf{G}) \to H^1(S_0, \mathbf{G}))$.

[1] Or more precisely, under the affine R_n-group scheme $\mathbf{Aut}(\mathfrak{g} \otimes_k R_n)$. This abuse of notation will be used throughout.

3. Statement of the isotriviality theorem: applications

Theorem 3.1. *If k is algebraically closed, then $H^1(R_{n,\infty}, \mathbf{G}) = 1$ for any linear algebraic group $\mathbf{G}/k$.*

We shall outline the proof of the Isotriviality Theorem later. Here are some applications.

Corollary 3.2. *Assume k is algebraically closed.*

1. $\mathrm{Spec}(R_{n,\infty})$ *is the simply connected covering of* $\mathrm{Spec}(R_n)$. *Thus $\pi_1(R_n, 1)$ is the Galois group of the pro-covering $R_{n,\infty}/R_n$, namely*
$$\pi_1(R_n, 1) = \varprojlim_d \mathrm{Gal}(R_{n,d}/R_n) = \varprojlim_d (\mathbf{Z}/d\mathbf{Z})^n = \big(\widehat{\mathbf{Z}}\big)^n.$$
2. *Let μ be a twisted finite constant R_n-group. Then $\mu \times_{R_n} R_{n,\infty}$ is constant. Furthermore $H^1(R_{n,\infty}, \mu) = 1$.*
3. *Let $\mathbf{G}$ be a reductive R_n-group (in the sense of [10]). Then $\mathbf{G} \times_{R_n} R_{n,\infty}$ is split. Furthermore $H^1(R_{n,\infty}, \mathbf{G}) = 1$.*

In particular, $H^1(\pi_1(R_n, 1), \mathbf{F}(R_{n,\infty})) \xrightarrow{\sim} H^1(R_n, \mathbf{F})$ for every R_n-group $\mathbf{F}$ which is an extension of a reductive group by a twisted finite constant group. (The étale cohomology of $\mathbf{F}/R_n$ is thus given by the usual Galois cocycles.)

Proof. (1) For any finite group G, we have $H^1(R_{n,\infty}, G_k) = 1$. The theory of the algebraic fundamental group ([9], Exp. V.7 and Exp. V Proposition 5.5) shows then that $\mathrm{Spec}(R_{n,\infty}) \to \mathrm{Spec}(R)$ is the simply connected covering of $\mathrm{Spec}(R_n)$. (One can also obtain this result by means of the Künneth formula ([9], XIII.4.6).)

(2) The group μ is a twisted form (in the étale topology) of a constant finite R_n-group μ_0. Since $\mathbf{Aut}(\mu_0)$ comes from a linear algebraic k-group, the theorem shows that the group $\mu \times_{R_n} R_{n,\infty}$ is constant, and that $H^1(R_{n,\infty}, \mu) = 1$. (Since μ is isotrivial, one can also reason as in the constant case of (3) below.)

(3) By the structure theorem of Demazure, there exists a unique Chevalley group $\mathbf{G}_0/\mathbf{Z}$ such that $\mathbf{G}/R_n$ is a R_n-form of $\mathbf{G}_0$. Consider the (split) exact sequence of $\mathbf{Z}$-group schemes
$$1 \to \mathbf{G}_{0,ad} \to \mathbf{Aut}(\mathbf{G}_0) \to \mathbf{Out}(\mathbf{G}_0) \to 1,$$
as in [10] Exp. XXIV Theorem 1.3. We claim that both outside terms of the corresponding exact sequence of pointed sets
$$H^1(R_{n,\infty}, \mathbf{G}_{0,ad}) \to H^1\big(R_{n,\infty}, \mathbf{Aut}(\mathbf{G}_0)\big) \to H^1\big(R_{n,\infty}, \mathbf{Out}(\mathbf{G}_0)\big)$$
vanish. Indeed. The left H^1 vanishes because $\mathbf{G}_{0,ad} \times_{\mathbf{Z}} k$ is linear. On the other hand, since R_n is normal and Noetherian, any R_n-torsor under $\mathbf{Out}(\mathbf{G}_0)$ is isotrivial ([10] Exp. X.6). By (1) then, the map $H^1(R_n, \mathbf{Out}(\mathbf{G}_0)) \to H^1(R_{n,\infty}, \mathbf{Out}(\mathbf{G}_0))$ is trivial, hence $H^1(R_{n,\infty}, \mathbf{Out}(\mathbf{G}_0)) = \varinjlim H^1(R_{n,d}, \mathbf{Out}(\mathbf{G}_0)) = 1$.

From this it follows that $H^1(R_{n,\infty}, \mathbf{Aut}(\mathbf{G}_0)) = 1$. Since this set classifies $R_{n,\infty}$-forms of $\mathbf{G}_0$, the group $\mathbf{G} \times_{R_n} R_{n,\infty}$ is isomorphic to $\mathbf{G}_0 \times_{R_n} R_{n,\infty}$, hence split. □

Corollary 3.3. *Let $\mathbf{G}$ be a reductive R_n-group, and let X be an R_n-torsor under $\mathbf{G}$. There exists a positive integer d, and a finite Galois extension K/k so that the base change $R_n \to K \otimes_k R_{n,d}$ trivializes X. In particular, X is isotrivial (i.e. trivialized by a finite étale extension of R_n).*

Proof. Over an algebraically closure of k, Corollary 3.2(3) yields an $R_{n,\infty}$-algebra isomorphism $k[\mathbf{G}_0] \otimes_k R_{n,\infty} \simeq R_n[X] \otimes_{R_n} R_{n,\infty}$. Since $k[\mathbf{G}_0]$ is of finite type, the above isomorphism factors through some $R_{n,d}$ and some K as stated. □

Corollary 3.4. *Every reductive group scheme over R_n is isotrivial (i.e. splits over a finite étale extension of R_n).*

Corollary 3.5. *Let A be a finite dimensional algebra over k. Let $\mathcal{L}$ be an R_n-form of A, namely an R_n-algebra $\mathcal{L}$ such $\mathcal{L} \otimes_{R_n} S$ that $\mathcal{L} \otimes_{R_n} S \simeq_S A \otimes_k S$ for some fppf extension S of R_n (isomorphism of S-algebras). Then such an isomorphism takes place with $S = K \otimes_k R_{n,d}$ for some K and d as in Corollary 3.3 above.*

Proof. The R_n-isomorphism class of such algebra $\mathcal{L}$ corresponds to an element of $H^1(R_n, \mathbf{Aut}(A))$. By Theorem 3.1 then, after passing to an algebraic closure of k, we have $\mathcal{L} \otimes_{R_n} R_{n,\infty} \simeq A \otimes_k R_{n,\infty}$. Since A is finite dimensional, the above isomorphism factors through some $R_{n,d}$ and K as stated. $\square$

4. Proof of the isotriviality theorem

Proposition 4.1. *Let $\mathbf{G}/k$ be a linear algebraic group. If $H^1_{Zar}(\mathbb{A}^n_E, \mathbf{G}) = 1$ for all extensions E/k, then $H^1_{Zar}(R_n, \mathbf{G}) = 1$.*

Proof. Since $H^1_{Zar}(\mathbb{A}^n_{k(t)}, \mathbf{G}) = 1$, an easy gluing argument shows that the canonical map $H^1_{Zar}(\mathbb{A}^{n+1}, \mathbf{G}) \simeq H^1_{Zar}(\mathbb{A}^n \times_k \mathbb{A}^1, \mathbf{G}) \to H^1_{Zar}(\mathbb{A}^n \times \mathbb{G}_m, \mathbf{G})$ is surjective. This allows us to reason by induction on n to establish that $H^1_{Zar}(\mathbb{A}^r \times_k \mathbb{G}_m^{n-r}, \mathbf{G}) = 1$ for all $0 \leqslant r \leqslant n$. $\square$

Corollary 4.2. *If $\mathbf{G}^0_{red}$ is quasi-split, then $H^1_{Zar}(R_n, \mathbf{G}) = 1$.*

Proof. It suffices to show that $H^1_{Zar}(\mathbf{A}^n_k, \mathbf{G}) = 1$. For the group $\mathbf{G}^0_{red}$, this is Theorem C of [7]. The general case reduces to this as shown in page 104 of [2]. $\square$

Proposition 4.3. *Let $\mathbf{G}/k$ be a linear algebraic group. Then*

$$H^1_{Zar}(R_{n,\infty}, \mathbf{G}) = H^1(R_{n,\infty}, 1_n, \mathbf{G}).$$

Proof. It is immediate from the definitions that $H^1_{Zar}(R_{n,\infty}, \mathbf{G}) \subset H^1(R_{n,\infty}, 1_n, \mathbf{G})$. By Theorem 3.2 of [2], rationally trivial torsors are locally trivial. In other words, we have an exact sequence

$$1 \to H^1_{Zar}(R_{n,\infty}, \mathbf{G}) \to H^1(R_{n,\infty}, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G}),$$

so that is enough to prove the triviality of the map $H^1(R_{n,\infty}, 1, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G})$. This is done by induction on n. The map $H^1(R_{1,d}, 1_1, \mathbf{G}) \to H^1(R_{1,\infty}, \mathbf{G})$ is trivial. For given $\gamma \in H^1(R_{1,d}, 1_1, \mathbf{G})$, we can appeal to Harder's Lemma [6] and Florence's acyclicity ([3] Proposition 5.4), for the existence of some multiple d' of d for which the image of γ in $H^1(R_{1,d'}, \mathbf{G})$ can be extended to the full affine line. This image is therefore trivial (by Raghunathan–Ramanathan's theorem [8] in the reductive connected case, and [2] page 104 in general). So γ is trivial in $H^1(R_{1,\infty}, \mathbf{G})$.

For $n > 1$, let $B = \varinjlim_d k[t_n^{\pm 1/d}]$ and $E = \varinjlim_d k(t_n^{1/d})$. Let $p^* : R_{n,\infty} \to E \otimes_k R_{n-1,\infty}$ be the natural map, and consider the following diagram of base change maps

$$\begin{array}{ccc} H^1(R_{n,\infty}, \mathbf{G}) & \xrightarrow{p_*} & H^1(E \otimes_k R_{n-1,\infty}, \mathbf{G}) \\ {\scriptstyle ev_{1_{n-1}}}\downarrow & & {\scriptstyle ev_{1_{n-1}}}\downarrow \\ H^1(B, \mathbf{G}) & \longrightarrow & H^1(E, \mathbf{G}). \end{array}$$

We have $ev_{1_{n-1}}\big(H^1(R_{n,\infty}, 1_n, \mathbf{G})\big) \subset H^1(B, 1_B, \mathbf{G})$, so the case $n = 1$ implies that $p_*(H^1(R_{n,\infty}, 1_n, \mathbf{G})) \subset H^1(E \otimes_k R_{n-1,\infty}, 1_{n-1}, \mathbf{G})$. We apply then the induction hypothesis to $E \otimes_k R_{n-1,\infty}$ with fraction field $K_{n,\infty}$ to conclude that the map

$$H^1(E \otimes_k R_{n-1,\infty}, \mathbf{G}) \to H^1(K_{n,\infty}, \mathbf{G})$$

is trivial. It now follows that $H^1(R_{n,\infty}, 1_n, \mathbf{G})$ consists of rationally trivial torsors. $\square$

Assume now that k is algebraically closed. Then $H^1(R_{n,\infty}, \mathbf{G}) = H^1(R_{n,\infty}, 1_n, \mathbf{G})$, so that $H^1_{Zar}(R_{n,\infty}, \mathbf{G}) = H^1(R_{n,\infty}, \mathbf{G})$ by Proposition 4.3. The group $\mathbf{G}^0_{red}$ is split, so $H^1_{Zar}(R_n, \mathbf{G}) = 1$ by Corollary 4.2. It follows that $H^1_{Zar}(R_{n,\infty}, \mathbf{G}) = 1$, hence that $H^1(R_{n,\infty}, \mathbf{G}) = 1$. This establishes Theorem 3.1.

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