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Differential Topology

A note on logarithmic transformations on the Hopf surface

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Abstract

In this note we study logarithmic transformations in the sense of differential topology on two fibers of the Hopf surface. It is known that such transformations are susceptible to yield exotic smooth structures on 4-manifolds. We will show here that this is not the case for the Hopf surface, all integer homology Hopf surfaces we obtain are diffeomorphic to the standard Hopf surface. ***To cite this article: R. Zentner, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Transformations logarithmiques sur la surface de Hopf. Dans cette note nous étudions des transformations logarithmiques au sens de la topologie différentielle le long de deux fibres de la surface de Hopf. Il est connu que ce type de transformations peut donner lieu à des structures différentiables exotiques sur les 4-variétés. Nous allons montrer que ceci n'est pas le cas pour la surface de Hopf. En effet, les surfaces de Hopf homologiques que nous obtenons sont difféomorphes à la surface de Hopf standard. ***Pour citer cet article : R. Zentner, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Définition 0.1. Soit $\pi : X \to \Sigma$ une fibration elliptique. On dit que la 4-variété X' est obtenue à partir de X par une transformation logarithmique sur la fibre régulière F si X' est le résultat de l'opération suivante : On enlève un voisinage tubulaire νF de F et on colle $T^2 \times D^2$ à $X - \nu F$ par un difféomorphisme $\varphi : T^2 \times S^1 \to \partial \nu F$.

La valeur absolue du degré de $\pi|_{\partial\nu F} \circ \varphi|_{pt \times S^1}$ s'appelle la multiplicité de la transformation logarithmique. Le difféomorphisme φ est déterminé, à isotopie près, par l'isomorphisme entre les groupes fondamentaux qu'il induit. Considérons la 4-variété X' obtenue à partir de la surface de Hopf $X = S^1 \times S^3 \to S^2$ par transformations logarithmiques $\varphi_\pm$ sur deux fibres $F_\pm$. Si φ_+ (resp. φ_-) est de direction (a, b) (resp. (c, d)) et de multiplicité p (resp. q), alors X' aura pour groupe fondamental

$$\pi_1(X') = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = 1,\ [\alpha, \gamma] = 1,\ [\beta, \gamma] = 1, \alpha^a \beta^b (\alpha\gamma^{-1})^p = 1,\ \alpha^c \beta^d \gamma^q = 1 \rangle.$$

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On trouve donc que $\pi_1(X') \cong \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, où μ est le plus grand commun diviseur de tous les mineurs d'ordre 2 d'une matrice de présentation de $\pi_1(X')$. Pour de nombreux choix possibles μ sera égal à 1.

Notons maintenant par $X_\psi := (T^2 \times D^2) \cup_\psi (T^2 \times D^2)$ la 4-variéte qu'on obtient en collant $T^2 \times D^2$ à $T^2 \times D^2$ via le difféomorphisme ψ entre leurs bords, et soit ψ_* l'isomorphisme entre groupes fondamentaux induit. Nous obtenons ainsi la variété X' via le recollement

$$\left(T^2 \times D^2\right) \cup_{\varphi_+^{-1}} \left(T^2 \times A^2\right) \cup_\zeta \left(T^2 \times A^2\right) \cup_{\varphi_-} \left(T^2 \times D^2\right) \cong X_{\varphi_+^{-1} \circ \zeta \circ \varphi_-}.$$

Le difféomorphisme ζ donne la surface de Hopf standard : $X_\zeta = S^1 \times S^3$. En considérant l'automorphisme de groupe fondamental induit par $\varphi_+^{-1} \circ \zeta \circ \varphi_-$ on peut voir si la variété X' est une surface de Hopf homologique.

Théorème 0.2. *Supposons que la variété X_ψ est une surface de Hopf homologique. Alors X_ψ est difféomorphe à la surface de Hopf standard $X_\zeta = S^1 \times S^3$.*

Corollaire 0.3. *Si deux transformations logarithmiques le long de deux fibres de la surface de Hopf résultent dans une surface de Hopf homologique, alors cette variété est difféomorphe à la surface de Hopf standard $S^1 \times S^3$.*

La démonstration du Théorème 0.2 utilise le fait que les variétés X_ψ et $X_{\psi_t \circ \psi \circ \psi_b}$ sont difféomorphes, si ψ_t et ψ_b sont des difféomorphismes de $T^2 \times S^1$ qui se prolongent en tant que difféomorphisme de $T^2 \times D^2$. Ceci permet certaines opérations sur les lignes et les colonnes de $\psi_* \in \mathrm{Sl}(3, \mathbb{Z})$. On obtient ainsi une certaine forme standard pour ψ_* si X_ψ est une surface de Hopf homologique. Ces possibilités pour ψ_* se distinguent par des élements de $\mathrm{Sl}(2, \mathbb{Z})$. En utilisant un argument sur l'attachement de 2-anses, on observe finalement que toutes ces matrices ψ_* induisent des variétés difféomorphes.

1. Introduction

The (standard) Hopf surface $S^1 \times S^3$ fibers over the 2-sphere S^2 via the map obtained by composing the Hopf fibration $S^3 \to S^2$ with the projection on the second factor. Any fibre is diffeomorphic to the torus T^2 and there are no singular fibers, because this map is a submersion. It is a natural problem to study the effect of logarithmic transformations on two fibres in this case. Indeed, this operation was successfully used in the case of the K3 surface to construct exotic K3 manifolds as well as on other elliptic fibrations. These results have been obtained using gauge theoretical methods, which only apply for manifolds with $b_2^+ \geqslant 1$ [1,3,8]. Note that all K3-surfaces are diffeomorphic 4-manifolds, and there exist complex K3-surfaces which are elliptic fibrations. In the case of the K3-surface the resulting manifolds depend only on the multiplicities of the logarithmic transformations, but in our considerations they depend on some additional parameters as well.

For 4-manifolds with the rational homology of a Hopf surface the existing gauge theoretical methods do not apply. On the other hand it is a fundamental and open problem whether 4-manifolds with $b_2 \leqslant 4$ [9] (like the 4-sphere and the Hopf surface) do admit exotic structures. In the complex geometric framework, exotic Hopf surfaces do not exist, for by a result of Kodaira [6] every complex surface which is homeomorphic to $S^1 \times S^3$ is a primary Hopf surface, so it is diffeomorphic to $S^1 \times S^3$. Complex surfaces which are rational homology Hopf surfaces have been classified in [2] using logarithmic transformations. Further results about elliptic surfaces in the class of complex surfaces can be found in [3]; in particular, from Chapter 2.7 therein it follows that elliptic surfaces of Euler number zero with Abelian fundamental group, and which are integer homology Hopf surfaces, are the standard Hopf surface. Our considerations here, however, are purely topological in nature and the logarithmic transformations considered are more general than the complex-geometric ones. In particular, logarithmic transformations with multiplicity zero do not arise in the complex geometric setting, and may even result in manifolds not admitting any complex structure at all [4].

We will first calculate the fundamental group of the manifold obtained by two logarithmic transformations. As it will turn out, in many cases, including multiplicity 0, the resulting manifold will have the same fundamental group as the Hopf surface. Since the Euler characteristic is invariant under logarithmic transformations, we will obtain a manifold having the same (integer) homology as the Hopf surface. We will then describe a procedure to construct all these manifolds by gluing two copies of $T^2 \times D^2$ via a diffeomorphism between their boundaries. Using diffeomorphisms of $T^2 \times S^1$ which extend over $T^2 \times D^2$, we will be able to show that manifolds given by different gluing diffeomorphisms may still be diffeomorphic. Using this observation, we will find a certain standard form for every homology Hopf surface obtained by this gluing method. The possible standard forms are determined by elements in $\mathrm{Sl}(2,\mathbb{Z})$. Finally, using a handlebody-theoretical argument [7], we prove that this parameter does not affect the diffeomorphism type.

2. Logarithmic transformations applied to Hopf surfaces and resulting fundamental group

Definition 2.1. Let $\pi : X \to \Sigma$ be an elliptic fibration. We say that a 4-manifold X' is obtained from X by logarithmic transformation on a regular fibre F of π if X' is obtained from X through the following construction: We cut out a regular neighbourhood νF of F and we glue in a $T^2 \times D^2$ via an arbitrary orientation-reversing diffeomorphism $\varphi : T^2 \times S^1 \to \partial \nu F$. The absolute value of the degree of $\pi|_{\partial\nu F} \circ \varphi|_{pt\times S^1}$ is called the multiplicity of the logarithmic transformation [4].

The diffeomorphism φ is determined, up to isotopy, by its induced isomorphism of fundamental groups, which itself, after the choice of some bases, is determined by a matrix in $\mathrm{Gl}(3,\mathbb{Z})$. Alternatively, we fix one such diffeomorphism, which can be used to identify $\partial\nu F$ with $T^2 \times S^1$. Then any other is determined by a self-diffeomorphism of $T^2 \times S^1$, and these diffeomorphisms are given, up to isotopy, by elements in $\mathrm{Sl}(3,\mathbb{Z})$.

We will first give a gluing description of the Hopf-surface $X = S^1 \times S^3$ which will turn out useful in what follows. For this we shall first describe S^3 as two solid tori $S^1 \times D^2$ glued together. The two closed discs D^2 will turn out to be the northern respectively southern hemisphere under the Hopf fibration $S^3 \to S^2$. Indeed, S^3 can be seen as the following set:

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}.$$

The Hopf fibration is then given by the map $S^3 \to \mathbb{CP}^1$ given by $(z,w) \mapsto [z:w]$, and $\mathbb{CP}^1$ is diffeomorphic to S^2. Define S^3_+ to be the set of elements (z,w) such that $0 \leqslant |w|^2 \leqslant 1$, and S^3_- to be the set of elements (z,w) with $0 \leqslant |z|^2 \leqslant 1$. Then there are diffeomorphisms

$$S^3_+ \xrightarrow{f_+} S^1 \times D^2; \quad (z,w) \xmapsto{f_+} \left(\frac{z}{|z|}, \frac{w}{z}\right), \quad \text{and} \quad S^3_- \xrightarrow{f_-} S^1 \times D^2; \quad (z,w) \xmapsto{f_-} \left(\frac{w}{|w|}, \frac{z}{w}\right).$$

When we restrict $f_+ \circ f_-^{-1}$ to the boundary, then the map $\partial(S^1 \times D^2) \to \partial(S^1 \times D^2)$ is given by the formula

$$f_+ \circ f_-^{-1}(u,\xi) = (u\xi, \bar{\xi}).$$

We extend this latter map to the trivial S^1 factor by the identity, so that we get a map $\zeta : T^2 \times \partial D^2 \to T^2 \times \partial D^2$, $(u,v,\xi) \mapsto (u\xi, v, \bar{\xi})$. Here and further down (u,v) denotes an element in the fibre $T^2 = S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}$, whereas ξ denotes an element in the base $D^2 \subseteq \mathbb{C}$. We then get the description of the Hopf surface as a gluing

$$X = (T^2 \times D^2) \cup_\zeta (T^2 \times D^2). \tag{1}$$

Now let us consider the manifold X' obtained from the Hopf surface when performing logarithmic transformations on two fibres, say on the fibre F_+ over the north pole $x_+ := [1:0]$ and the fibre F_- over the south pole $x_- = [0:1]$, associated with diffeomorphisms $\varphi_\pm$. There are natural identifications of $\partial(X - \nu F_\pm)$ with the 'inner' boundary of $T^2 \times (D^2 - \mathring{D}^2_{1/2})$ according to the decomposition (1). Therefore the orientation-reversing diffeomorphisms $\varphi_\pm$ can be seen as an orientation-preserving diffeomorphism of $T^2 \times S^1$, because the above 'inner' boundary is with opposite orientation to the 'outer' boundary. Let us denote by $X_\pm$ the two manifolds $(T^2 \times (D^2 - \mathring{D}^2_{1/2})) \cup_{\varphi_\pm} (T^2 \times D^2)$. What a gluing of two manifold along the boundary really means is actually an identification of collar neighbourhoods of the boundaries of the two manifolds. In our case, this description is given as

$$X_\pm = (T^2 \times (D^2 - D^2_{1/3})) \cup_{\phi_\pm} (T^2 \times \mathring{D}^2_{2/3}),$$

where $\phi_\pm : (\frac{1}{3}, \frac{2}{3}) \times T^2 \times S^1 \to (\frac{1}{3}, \frac{2}{3}) \times T^2 \times S^1$ is given by $\phi_\pm(r,u,v,\xi) := (\frac{2}{3} - r, \varphi_\pm(u,v,\xi))$.

Let us now fix some paths inside $D^2 \times T^2$, where the disc is thought of a subset of $\mathbb{C}$, centered at the origin. Fix some base-point $(u_0, v_0, \xi_0) \in T^2 \times D^2$, where $|\xi_0| = \frac{1}{2}$, so that the base point is in the 'gluing area'. Let us define three paths $\alpha_\pm, \beta_\pm, \gamma_\pm$ by the formulae $\alpha_\pm(t) = (u_0, v_0\, \mathrm{e}^{\mathrm{i}t}, \xi_0)$, $\beta_\pm(t) = (u_0\, \mathrm{e}^{\mathrm{i}t}, v_0, \xi_0)$, and $\gamma_\pm(t) = (u_0, v_0, \xi_0\, \mathrm{e}^{\mathrm{i}t})$. The path $\gamma_\pm$ is then a meridian to the fibre $T^2 \times \{0\}$ over $x_\pm$, that is its projection onto the fibre is trivial, whereas $\alpha_\pm$ and $\beta_\pm$ induce a basis of the fundamental group of the fibre. Note that by the same formulae we can define paths $(\alpha'_\pm, \beta'_\pm, \gamma'_\pm)$ inside the pieces $T^2 \times D^2$ to be glued in with $\varphi_\pm$. Then $(\alpha_\pm, \beta_\pm, \gamma_\pm)$ induce a basis of $\pi_1(X - \nu F_\pm)$ and $(\alpha'_\pm, \beta'_\pm, \gamma'_\pm)$ induce a basis of $\pi_1(\partial(T^2 \times D^2))$. The diffeomorphisms $\varphi_\pm$ are then determined by their maps of fundamental groups

$$\varphi_*^+ = \begin{pmatrix} * & * & a \\ * & * & b \\ * & * & p \end{pmatrix}, \qquad \varphi_*^- = \begin{pmatrix} * & * & c \\ * & * & d \\ * & * & q \end{pmatrix},$$

which are elements in $\mathrm{Sl}(3, \mathbb{Z})$. The entries marked as $*$ will not be relevant to the fundamental group, as we shall see now. We call $(a, b) \in \mathbb{Z}^2$ the direction of the logarithmic transformation φ_+, and $|p|$ is its multiplicity.

In order to compute the fundamental group of X' we shall first compute the fundamental groups of $X_\pm$ and then glue them together via ζ. X_+ is given as the union of two open sets, namely the sets $X_1 = T^2 \times (D^2 - D^2_{1/3})$ and $X_2 = T^2 \times \mathring{D}^2_{2/3}$, with intersection $X_0 = T^2 \times (\mathring{D}^2_{2/3} - D^2_{1/3})$. Only, X_0 injects into X_1 via the natural inclusion i, but into X_2 via ϕ_+. The fundamental group of each piece is

$$\pi_1(X_0) = \langle \alpha_0, \beta_0, \gamma_0 \mid [\ ,\ ] = 1 \rangle, \quad \pi_1(X_1) = \langle \alpha, \beta, \gamma \mid [\ ,\ ] = 1 \rangle, \quad \text{and} \quad \pi_1(X_2) = \langle \alpha', \beta' \mid [\ ,\ ] = 1 \rangle.$$

By $[\ ,\ ]$ we simply mean that all commutator relations are satisfied. Now the Seifert–van Kampen theorem states that $\pi_1(X_+)$ has as generators together the ones of $\pi(X_1)$ and $\pi(X_2)$, all relations of $\pi_1(X_1)$ and of $\pi_1(X_2)$, and the additional relations

$$i(\alpha_0) = \phi(\alpha_0) \ \Leftrightarrow\ \alpha' = \phi(\alpha_0), \quad i(\beta_0) = \phi(\beta_0) \ \Leftrightarrow\ \beta' = \phi(\beta_0), \quad \text{and} \quad i(\gamma_0) = \phi(\gamma_0) \ \Leftrightarrow\ 1 = \phi(\gamma_0).$$

The first two relations imply that we can just drop the generators α' and β' together with these two relations. Therefore the fundamental group is $\pi_1(X_+) = \langle \alpha_+, \beta_+, \gamma_+ \mid [\ ,\ ] = 1,\ \alpha_+^a \beta_+^b \gamma_+^p = 1 \rangle$.

Correspondingly we get $\pi_1(X_-) = \langle \alpha_-, \beta_-, \gamma_- \mid [\ ,\ ] = 1,\ \alpha_-^c \beta_-^d \gamma_-^q = 1 \rangle$. In order to compute the fundamental group of $X' = X_+ \cup_\zeta X_-$ we proceed in the same way. T^2 times a 'middle annulus' injects into X_- via the natural inclusion, whereas it injects into X_+ via ζ. As we have $\zeta_*(\alpha_0) = \alpha_+$, $\zeta_*(\beta_0) = \beta_+$ and $\zeta_*(\gamma_0) = \alpha_+ \gamma_+^{-1}$ we get a final formula:

$$\pi_1(X') = \langle \alpha, \beta, \gamma \mid [\ ,\ ] = 1,\ \alpha^a \beta^b \left(\alpha\gamma^{-1}\right)^p = 1,\ \alpha^c \beta^d \gamma^q = 1 \rangle.$$

By the classification of finitely generated Abelian groups we find that we have an isomorphism $\pi_1(X') \cong \mathbb{Z} \oplus \mathbb{Z}/\mu\mathbb{Z}$, where μ is the highest common divisor of all the 2-minors of a presentation matrix for this group. It is easy to see that there are various choices possible for which this number equals 1, including cases where one of the multiplicities, or both of them, may be zero.

Remark 1. If we perform the two logarithmic transformations such that they are trivial on the S^1-factor, then the construction is S^1 times Dehn-surgery on the Hopf-link in S^3. The resulting 4-manifold is then S^1 times a lens space; this can be seen using the surgery description of lens spaces [5].

3. Formulation in terms of gluing two copies of $T^2 \times D^2$

We will denote by $X_\varphi := (T^2 \times D^2) \cup_\varphi (T^2 \times D^2)$ the 4-manifold obtained by gluing $T^2 \times D^2$ to $T^2 \times D^2$ via the orientation-reversing diffeomorphism φ between their boundaries. Let us denote by A^2 an annulus. There are canonical identifications of the boundary-components of $T^2 \times A^2$ with $T^2 \times S^1$, as before.

We will show here that all of the manifolds considered so far can be obtained by gluing just two copies of $T^2 \times D^2$ along their boundaries:

Lemma 3.1. *We have the following diffeomorphism*: $X_{\psi\circ\varphi} \cong (T^2 \times D^2) \cup_\psi (T^2 \times A^2) \cup_\varphi (T^2 \times D^2)$.

Proof. As any diffeomorphism of one boundary-component of $T^2 \times A^2$ extends over the whole of $T^2 \times A^2$ the result follows easily. □

Our next purpose is to calculate the fundamental group of X_φ. Let us use the bases $(\alpha_\pm, \beta_\pm, \pm\gamma_\pm^{\pm 1})$ from above (up to 'orientation') and suppose that the map φ_*, which is now given by an element of $\mathrm{Sl}(3,\mathbb{Z})$, looks as follows:

$$\varphi_* = \begin{pmatrix} a & c & g \\ b & d & h \\ e & f & k \end{pmatrix}. \tag{2}$$

By the Theorem of Seifert–van Kampen a presentation of the fundamental group of X_φ is given by $\pi_1(X_\varphi) = \langle \alpha, \beta \mid [\alpha,\beta] = 1,\ (\alpha^{g'}\beta^{h'})^{(g,h)} = 1 \rangle$. Here (g,h) denotes the greatest common divisor of g and h, and g', h' are such that $g = (g,h)\ g'$, $h = (g,h)\ h'$. We set $(0,0) := 0$. The fundamental group is therefore isomorphic to $\pi_1(X_\varphi) = \mathbb{Z} \oplus \mathbb{Z}/(g,h)\mathbb{Z}$. In particular, X_φ is a homology Hopf surface if and only if $(g,h) = 1$, noting that any X_φ has Euler-characteristic zero.

If now we perform the logarithmic transformations associated with $\varphi_\pm$ on the two fibers $F_\pm$ of the Hopf surface, then the resulting manifold will be given by the following gluing construction

$$\left(T^2 \times D^2\right) \cup_{\varphi_+^{-1}} \left(T^2 \times A^2\right) \cup_\zeta \left(T^2 \times A^2\right) \cup_{\varphi_-} \left(T^2 \times D^2\right)$$

which is diffeomorphic, by the above lemma, to $X_{\varphi_+^{-1}\circ\zeta\circ\varphi_-}$. Whether this manifold is a homology Hopf surface can now be read off from the automorphism $(\varphi_+^{-1} \circ \zeta \circ \varphi_-)_*$ of the fundamental group. However, calculating by this matrix product the entity (g,h), which a posteriori depends on the numbers $a,\ b,\ p$ and $c,\ d,\ q$ only, is a rather hard problem.

Theorem 3.2. *Suppose the manifold $X_\varphi := (T^2 \times D^2) \cup_\varphi (T^2 \times D^2)$, obtained from gluing with the orientation-reversing diffeomorphism φ, is a homology Hopf surface. Then X_φ is diffeomorphic to the Hopf surface $S^1 \times S^3$.*

Corollary 3.3. *If logarithmic transformations on two fibers yield a homology Hopf surface then this 4-manifold is diffeomorphic to the standard Hopf surface $S^1 \times S^3$.*

Proof of the theorem. Observe first that the two manifolds X_φ and $X_{\psi_t^{-1}\circ\varphi\circ\psi_b}$ are diffeomorphic as soon as the diffeomorphisms ψ_t and ψ_b of $T^2 \times S^1$ extend over $T^2 \times D^2$ as diffeomorphisms. A diffeomorphism ψ extends iff the associated matrix has the form

$$\psi_* = \begin{pmatrix} r & t & 0 \\ s & u & 0 \\ v & w & 1 \end{pmatrix}. \tag{3}$$

This observation can be used to commit certain line operations on φ_* by left-multiplication with matrices induced by extending diffeomorphisms, as well as to commit certain column operations by right-multiplication with these matrices, and this without changing the diffeomorphism type.

Suppose now that X_φ is a homology Hopf surface with associated matrix φ_* as in (2) above. In particular, the greatest common divisor of g and h is one: $(g,h) = 1$. By left-multiplying with a matrix $U \in \mathrm{Sl}(2,\mathbb{Z}) \subseteq \mathrm{Sl}(3,\mathbb{Z})$, where the inclusion is as the upper left part in the 3×3 matrix, we may assume that $g = 1, h = 0$ in (2). Such a matrix U is of type (3). Now there is a matrix L of type (3) such that left-multiplication of the new matrix φ_* by L adds $-(k-1)$ times the first line of φ_* to its last line. Therefore we may suppose that $k = 1$. Now there is a matrix R of the type (3) such that right-multiplication of the newest φ_* by R will add appropriate multiples of the third column of φ_* to its first and second, so that we may assume $e = f = 0$ because $k = 1$. φ_* in (2) may therefore be supposed to have the form

$$\varphi_* = \begin{pmatrix} a & c & 1 \\ b & d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

A corresponding diffeomorphism is given by $\varphi(u,v,z) = (u^a v^c z, u^b v^d, z)$. Now we cannot simplify much further in order to obtain the matrix ζ_*, where ζ is inducing the standard Hopf surface as above. However, the attaching of $T^2 \times D^2$ to the upper $T^2 \times D^2$, which we shall denote by X_+, may be done by attaching first a 2-handle, then two 3-handles, and eventually a 4-handle. To be more precise, decompose the torus T^2 in the obvious way into a

0-handle Σ_0, two 1-handles Σ_{11} and Σ_{12}, and a 2-handle Σ_2. Then the attaching, via φ, of $\Sigma_0 \times D^2$ to X_+ is done along $\Sigma_0 \times \partial D^2$, so we attach a 2-handle and get $X^{(2)} := X_+ \cup (\Sigma_0 \times \partial D^2)$. It is now easily checked that $\Sigma_{11} \times D^2$ and $\Sigma_{12} \times D^2$ are attached to $X^{(2)}$ along a thickened 2-sphere $S^2 \times D^1$, so their attaching corresponds to 3-handle attachment. Finally $\Sigma_2 \times D^2$ is glued to the resulting manifold along a 3-sphere, so that this corresponds to a 4-handle attachment. Now the union of the two 3- and the 4-handle is diffeomorphic to a boundary sum $S^1 \times D^3 \natural S^1 \times D^3$, which is the gluing of two pieces of $S^1 \times D^3$ via a diffeomorphism between two discs in their boundaries. The boundary of this manifold is $S^1 \times S^2 \# S^1 \times S^2$. It is known [7] that any diffeomorphism of $S^1 \times S^2 \# S^1 \times S^2$, extends over the whole boundary sum. Therefore only the 2-handle-attachment is relevant for determining the diffeomorphism type of the closed 4-manifold.

On the other hand, the attaching of $\Sigma_0 \times \partial D$ is determined, up to isotopy, by the attaching of the attaching sphere $\{0\} \times S^1$ as well as the isomorphism of normal bundles $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1) \to \nu_{T^3}(\varphi(\{0\} \times S^1))$ induced by the derivative $d\varphi$. We shall denote by L_φ this bundle isomorphism. After identification of Σ_0 with a ball centered in the origin in $\mathbb{R}^2$ we get a canonical isomorphism $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1) \cong S^1 \times \mathbb{R}^2$. By a framing f of $\varphi(\{0\} \times S^1)$ we understand a fixed isomorphism of the normal bundle $\nu_{\Sigma_0 \times S^1}(\{0\} \times S^1)$ with $S^1 \times \mathbb{R}^2$. We say that a framing f is isotopic to the framing f' if they are homotopic through bundle isomorphisms. By replacing L_φ with f^{-1} we see that the 2-handle attachment is determined by $(\varphi(\{0\} \times S^1), f)$, the embedding of the attaching sphere and a framing for it. So framings and the isomorphisms L_φ are equivalent notions. Up to isotopy the attachment depends only on the framing up to isotopy. If we fix one framing, we see that all possible isomorphisms of normal bundles are given by bundle automorphisms of $S^1 \times \mathbb{R}^2$.

Now for the above choice of φ the attaching of the attaching sphere does not depend on the specific entries in φ_*. We identify the normal bundle of $\varphi(\{0\} \times S^1)$ with orthogonal complement to its tangent bundle within $T(T^3)$, and get an identification with $S^1 \times \mathbb{R}^2$ by specifying two constant orthonormal sections of that bundle, $e_1 = (1,\ 0,\ -1)$ and $e_2 = (0,\ 1,\ 0)$. The isomorphism L_φ is then given by the *constant* matrix

$$L_\varphi = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Because this matrix is in $\mathrm{Sl}(2, \mathbb{Z})$ we see that there is an isotopy of bundle automorphisms taking one into the other, in other words the corresponding framings are isotopic. □

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