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Differential Geometry

Minimal singular Riemannian foliations ☆

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Abstract

We prove that a singular foliation on a compact manifold admitting an adapted Riemannian metric for which all leaves are minimal must be regular. ***To cite this article: V. Miquel, R.A. Wolak, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Feuilletages Riemanniens singuliers. Nous prouvons que tout feuilletage singulier sur une variété compacte qu'a une métrique riemannienne feuilletée avec feuilles minimales est régulier. ***Pour citer cet article : V. Miquel, R.A. Wolak, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let M be a smooth manifold. A generalized differentiable distribution $\mathcal{D} \subset TM$ (i.e. $\dim \mathcal{D}_x$ $x \in M$ is not constant) is called a foliation if it is completely integrable, i.e. if at any point x of M it admits a maximal integral submanifold, called a leaf s of the foliation $\mathcal{D}$. The Sussmann–Stefan–Frobenius theorem provides necessary and sufficient conditions for complete integrability of generalized differentiable distributions, cf. [6]. A foliation $\mathcal{F}$ on a smooth manifold M (cf. [3], p. 189) is a *Riemannian foliation* (RF for short) if there is a Riemannian metric on M adapted to $\mathcal{F}$ in the sense that every geodesic which is orthogonal to a leaf at one point remains orthogonal to every leaf it meets. A Riemannian foliation $\mathcal{F}$ is called *regular* (RRF) if all the leaves have the same dimension. Otherwise, it is called *singular* (SRF). In an SRF, the leaves of maximal dimension are called regular, and the others are called singular leaves.

A foliation $\mathcal{F}$ on a manifold M is *taut* if there is a metric g on M such that every leaf of $\mathcal{F}$ is minimal. The study of these foliations was stimulated by Haefliger's paper cf. [1] in which the author demonstrated that in the regular case 'tautness' is a transverse property. In 1983, Carrière proposed the following conjecture for compact manifolds: *A RRF is taut if and only if the basic cohomology* $H^{\operatorname{codim}\mathcal{F}}(M/\mathcal{F}) \neq 0$. The problem was finally solved by Masa,

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cf. [2], in the early nineties. In the case of RRF the minimal metric can be chosen to be adapted to the foliation, cf. [5], p. 96. In fact, Ph. Tondeur demands the existence of an adapted metric in the definition of 'taut'. Therefore it is natural to formulate the following question: *"Do there exist 'taut' singular Riemannian foliations, and if so, can they be characterized cohomologically?"*. In this Note we prove that the singular case is very different from the regular one.

Theorem 1.1. *Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M. Then, there is no Riemannian metric on M adapted to $\mathcal{F}$ for which the leaves of $\mathcal{F}$ are minimal.*

2. Preliminaries

Given a SRF $\mathcal{F}$ on M, we shall fix an adapted Riemannian metric $\langle , \rangle$ on M. If $x \in M$, then L_x is the leaf of $\mathcal{F}$ passing through the point x. Given any leaf L of $\mathcal{F}$ and a connected open set P in L, we shall denote by P_r (resp. ∂P_r) the tube (resp. the tubular hypersurface) of radius r centered at P, that is

$$P_r = \{\exp tu;\ u \in N_x P,\ x \in P,\ 0 \leqslant t < r\}, \qquad \partial P_r = \{\exp ru;\ u \in N_x P,\ x \in P\}, \tag{1}$$

where $N_x P$ denotes the unit sphere fiber at x of the normal bundle of P, and exp denotes the exponential map for the metric $\langle , \rangle$. Crucial for our result is the following structure theorem of a SRF:

Theorem 2.1 (Homothety Lemma, [3], p. 193). *Given $x_0 \in M$ compact, let P be a relatively compact connected open neighborhood of x_0 in the leaf L_{x_0} through x_0. Then, there is a $\rho_0 > 0$ such that,*

(i) *for every $x \in P_{\rho_0}$, the connected component P_x of x in $L_x \cap P_{\rho_0}$ (called a plaque of F through x) is contained in $\partial P_{\mathrm{dist}(P,x)}$, and*
(ii) *for $\lambda > 0$ and $\rho > 0$ such that ρ and $\lambda\rho$ are both $< \rho_0$, the diffeomorphism $\partial P_\rho \to \partial P_{\lambda\rho}$ defined by* $\exp \rho u \mapsto \exp \lambda\rho u$ *sends one plaque onto one plaque.*

Moreover, we shall use the well known first variation formula for the volume of a submanifold R of M when we take a normal variation $R_t = \{\exp_y tX_y;\ y \in R\}$ of R (X being a vector field normal to R called variation vector field)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{volume}(R_t)\big|_{t=0} = \int_R \langle H_y, X_y \rangle \eta, \tag{2}$$

where η is the volume element of R and H is the mean curvature vector of R.

3. Proof of Theorem 1.1

The idea of the proof is the following: If $\mathcal{F}$ is a SRF, the Homothety Lemma (Theorem 2.1) implies that, in a suitable neighborhood of an open set of a singular leaf L of dimension q, the regular leaves R of dimension $q + k$ look like tubular submanifolds around L, with spherical shaped slices of dimension k (the vectors $Z_{q+j}(s)$ in the formula (4) below are tangent to these slices). Then, in the direction from R to L, and near L, the contribution of these spherical shaped slices makes the volume of the regular leave to decrease (this is the meaning of formula (12) below). Since the mean curvature gives the variation of the volume of the regular leaves (formula (2)), it cannot be zero. This incompatibility with minimality disappears only when there are no spherical shaped slices, that is, when $k = 0$. Now, let us go into the details.

Let us suppose that $\mathcal{F}$ is a SRF. Let $q + k$ be the maximal dimension of the leaves of the foliation, and q the dimension of a singular leaf L such that, being $\mathcal{S}$ a relatively compact connected open set in L, in the tube $\mathcal{S}_{\rho_0}$, with the ρ_0 given in Theorem 2.1, there is a plaque $\mathcal{R}$ of a leaf of maximal dimension. From Theorem 2.1(i) it follows that $\mathcal{R}$ is at constant distance r from $\mathcal{S}$ and there is a subset $\mathcal{U}$ of the unit normal bundle $\mathcal{NS}$ of $\mathcal{S}$ such that $\mathcal{R} = \exp(r\mathcal{U})$.

Moreover, it is a consequence of Theorem 2.1(ii) that, for any $s \in (0, r)$, $\mathcal{R}_s = \exp(s\mathcal{U})$ is a plaque of a leaf of dimension $q + k$ of the foliation. The subset $\mathcal{U}$ can be written as

$$\mathcal{U} = \bigcup_{x \in \mathcal{S}} U_x, \quad U_x = \mathcal{U} \cap N_x\mathcal{S}.$$

Let us consider on $\mathcal{U}$ the volume form $\mathrm{d}x \wedge \mathrm{d}u_x$, where $\mathrm{d}x$ is the volume form of $\mathcal{S}$ and $\mathrm{d}u_x$ is the volume form of U_x. As $\mathcal{R}_s = \exp(s\mathcal{U})$, we have the following diffeomorphism:

$$\varphi : \mathcal{U} \subset \mathcal{N}\mathcal{S} \longrightarrow \mathcal{R}_s \subset M \quad \text{defined by } U_x \ni u_x \mapsto \exp s u_x.$$

Then the volume form η_s of $\mathcal{R}_s$ has the pullback $\varphi^*\eta_s$ that can be written as

$$\varphi^*\eta_s = \phi(s, x, u_x)\, \mathrm{d}x \wedge \mathrm{d}u_x, \tag{3}$$

where ϕ can be computed, taking orthonormal basis $\{e_1, \ldots, e_q\}$ on $T_x\mathcal{S}$ and $\{e_{q+1}, \ldots, e_{q+k}\}$ of $T_{u_x}U_x$ (recall that $k = \dim U_x = \dim \mathcal{R}_s - \dim \mathcal{S}$) as follows

$$\begin{aligned}\phi(s, x, u_x) &= \varphi^*\eta_s(e_1, \ldots, e_q, e_{q+1}, \ldots, e_{q+k}) \\ &= \eta_s(\exp_{*su_x} se_1, \ldots, \exp_{*su_x} se_q, \exp_{*su_x} se_{q+1}, \ldots, \exp_{*su_x} se_{q+k}) \\ &= \eta_s\big(Y_1(s), \ldots Y_q(s), Z_{q+1}(s), \ldots, Z_{q+k}(s)\big),\end{aligned} \tag{4}$$

where $Y_i(t), Z_{q+j}(t)$ (see, for instance [4], pages 36 and 58–60) are the Jacobi fields along the geodesic $t \mapsto \exp_x tu_x$, $0 \leqslant t \leqslant r$, satisfying

$$Y_i(0) = e_i, \qquad Y_i'(0) + L_{u_x} e_i \in (T_x\mathcal{S})^{\perp}, \tag{5}$$

where L_{u_x} is the Weingarten map of $\mathcal{S}$ in the direction of u_x,

$$Z_{q+j}(0) = 0, \quad Z'_{q+j}(0) = e_{q+j}, \quad \text{and} \tag{6}$$

$$\lim_{t\to 0} \frac{Z_{q+j}(t)}{t} = \lim_{t\to 0} \frac{1}{t} \exp_{*tu_x} te_{q+j} = e_{q+j}. \tag{7}$$

On the other hand, it follows from (3) that $\mathrm{volume}(\mathcal{R}_s) = \int_{\mathcal{R}_s} \eta_s = \int_{\mathcal{S}} \int_{U_x} \phi(s, x, u_x)\, \mathrm{d}x \wedge \mathrm{d}u_x$, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{volume}(\mathcal{R}_s) &= \int_{\mathcal{S}}\int_{U_x} \frac{\mathrm{d}}{\mathrm{ds}}\phi(s, x, u_x)\, \mathrm{d}x \wedge \mathrm{d}u_x = \int_{\mathcal{S}}\int_{U_x} \frac{\frac{\mathrm{d}}{\mathrm{ds}}\phi(s, x, u_x)}{\phi(s, x, u_x)}\, \phi(s, x, u_x)\, \mathrm{d}x \wedge \mathrm{d}u_x \\ &= \int_{\mathcal{R}_s} \frac{\frac{\mathrm{d}}{\mathrm{ds}}\phi(s, x, u_x)}{\phi(s, x, u_x)}\, \eta_s.\end{aligned} \tag{8}$$

On the other hand, when we apply formula (2) to the submanifold $\mathcal{R}_s$ and the variation vector field $\exp_x su_x \mapsto \frac{\mathrm{d}}{\mathrm{ds}}(\exp_x su_x)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{ds}}\mathrm{volume}(\mathcal{R}_s)|_{s=s} = \int_{\mathcal{R}_s} \left\langle H_{\exp_x su_x}, \frac{\mathrm{d}}{\mathrm{ds}}(\exp_x su_x)\right\rangle \eta_s, \tag{9}$$

$H_{\exp_x su_x}$ being the mean curvature of $\mathcal{R}_s$ at $\exp_x su_x$. Now, let $f : \mathcal{R}_s \to \mathbb{R}$ be the C^∞ function

$$f(\exp_x su_x) = \phi(s, x, u_x)^{-1} \frac{\mathrm{d}}{\mathrm{ds}}\phi(s, x, u_x) - \left\langle H_{\exp_x su_x}, \frac{\mathrm{d}}{\mathrm{ds}}(\exp_x su_x)\right\rangle.$$

Let $\mathcal{R}_s^+$ (resp. $\mathcal{R}_s^-$) be the set of points of $\mathcal{R}_s$ where f is > 0 (resp. < 0). The same arguments used to prove formulae (8) and (9) show that they are still true for $\mathcal{R}_s^+$ and $\mathcal{R}_s^-$. Then

$$\int_{\mathcal{R}_s^+} f(\exp_x su_x)\eta_s = 0 = \int_{\mathcal{R}_s^-} f(\exp_x su_x)\eta_s,$$

and these equalities imply that $\mathcal{R}_s^+ = \emptyset = \mathcal{R}_s^-$. Then, f is identically zero, that is, for every $\exp_x su_x \in \mathcal{R}_s$,

$$\phi(s, x, u_x)^{-1} \frac{\mathrm{d}}{\mathrm{ds}}\phi(s, x, u_x) = \left\langle H_{\exp_x su_x}, \frac{\mathrm{d}}{\mathrm{ds}}(\exp_x su_x)\right\rangle. \tag{10}$$

Let $\xi_1, \ldots, \xi_l,\ l = n - q - k$, be a family of unit orthogonal vector fields along $\exp_x tu_x$ which are orthogonal to $Y_1, \ldots, Z_{q+k}$. Then, for every $t \in [0, s]$,

$$\eta_s(Y_1, \ldots, Y_q, Z_{q+1}, \ldots, Z_{q+k}) = \omega(Y_1, \ldots, Y_q, Z_{q+1}, \ldots, Z_{q+k}, \xi_1, \ldots, \xi_l),$$

where ω is the volume form of M. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\phi(s, x, u_x) = {} & \sum_i \omega(Y_1, \ldots, Y_i', \ldots, Y_q, Z_{q+1}, \ldots, \xi_l) \\ & + \sum_j \omega(Y_1, \ldots Y_q, Z_{q+1}, \ldots, Z_{q+j}', \ldots, Z_{q+k}, \xi_1, \ldots, \xi_l) \\ & + \sum_\alpha \omega(Y_1, \ldots, Z_{q+k}, \xi_1, \ldots, \xi_\alpha', \ldots, \xi_l).\end{aligned}$$

Since $\langle \xi_\alpha, \xi_\alpha \rangle = 1$, the vector fields ξ_s' and ξ_s are orthogonal. Therefore, the last term of the sum vanishes. Using this, we can write

$$\begin{aligned}\frac{\mathrm{d}\phi/\mathrm{d}s}{s^{k-1}} = {} & \sum_i \omega\left(Y_1, \ldots, Y_i', \ldots, Y_q, Z_{q+1}, \frac{Z_{q+2}}{s}, \ldots, \frac{Z_{q+k}}{s}, \xi_1, \ldots, \xi_l\right) \\ & + \sum_j \omega\left(Y_1, \ldots, Y_q, \frac{Z_{q+1}}{s}, \ldots, Z_{q+j}', \ldots, \frac{Z_{q+k}}{s}, \xi_1, \ldots, \xi_l\right).\end{aligned} \tag{11}$$

From (6), $Z_{q+1}(0) = 0$, then the first adding term in (11) goes to 0 as s tends to 0 (since the other vectors remain bounded). From (5), (6) and (7) the second adding term in (11) has limit 1 when s goes to 0, then

$$\lim_{s \to 0} s^{-(k-1)} \frac{\mathrm{d}\phi}{\mathrm{d}s} = 1, \tag{12}$$

and therefore, near $s = 0$, $\frac{\mathrm{d}}{\mathrm{d}s}\phi(s, x, u_x) \neq 0$. Thus, it follows from (10) that the mean curvature of this $\mathcal{R}_s$ is not identically 0.

We have started with a SRF $\mathcal{F}$ on M, fixed an arbitrary adapted Riemannian metric $\langle , \rangle$, picked an arbitrary singular leaf of dimension $q < q + k$ and showed that, in a neighborhood of it there are regular leaves whose mean curvature is not identically 0. So, the proof of Theorem 1.1 is finished.

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