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Numerical Analysis

Finite elements for a prefractal transmission problem

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Abstract

In this Note we deal with the finite element approximation of a transmission problem across a prefractal curve approximating the von Koch fractal curve. We construct a mesh adapted to the geometric shape of the interface and we refine it consistently with some estimates in suitable weighted Sobolev spaces. In these spaces we also obtain an approximation error estimate. ***To cite this article: P. Bagnerini et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Éléments finis pour un problème de transmission préfractale. Cette Note concerne l'approximation éléments finis d'un problème de transmission à travers la courbe préfractale approchant la courbe fractale de von Koch. On construit un maillage adapté à la géométrie de l'interface et on génère un processus de raffinement de maillage en utilisant des estimations dans des espaces de Sobolev à poids, choisis convenablement. On obtient aussi dans ces espaces une estimation de l'erreur d'approximation. ***Pour citer cet article : P. Bagnerini et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Several phenomena encountered in nature or in technical processes can be modelled by transmission problems with irregular interfaces. Fractal and prefractal geometries can be adopted as mathematical models for irregularity (see, e.g., [4]). We deal with a transmission problem in a polygonal domain, where the interface is a polygonal prefractal curve approximating the von Koch curve. This type of problem is firstly studied, from an analytical point of view, in [5] and in [6]. In this work, we formulate a Galerkin method on adapted meshes and we obtain an approximation error estimate in suitable weighted Sobolev spaces, depending on the regularity of the solution. The mesh refinement strategy and the error estimate are obtained in the spirit of [1]. We generate a sequence of 'nested meshes', one for each prefractal problem approximating the limit fractal problem, and we construct a mesh refinement algorithm, consistent with some estimates in weighted Sobolev spaces.

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2. Position of the problem

We consider a bounded convex polygonal domain Ω in $\mathbb{R}^2$: for example the open parallelogram with vertices $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1/2, \sqrt{3}/2)$, $P_4 = (1/2, -\sqrt{3}/2)$ as in Fig. 1. For every n in $\mathbb{N}$, let K_n be the n-th prefractal curve approximating the von Koch curve and having as endpoints P_1 and P_2. We briefly describe the procedure to construct the von Koch curve, which is an example of self-similar fractal. Let K_0 be a line segment of unit length having as endpoints P_1 and P_2. Let K_1 be obtained by dividing K_0 in three equal parts, removing the central segment and replacing it by the other two sides of the equilateral triangle based on the removed segment. Iterating this procedure to each of the segments of K_1, we construct a sequence of prefractal polygonal curves K_n which tends in a suitable sense to a limit curve K, called the von Koch curve, as n tends to infinity. As in Fig. 1, the interface K_n divides Ω in two subsets Ω_n^1 and Ω_n^2, the part of Ω which lies respectively over and under K_n.

We consider the n-th transmission problem

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n^i,\ i = 1,2, \\ -\mu_n \Delta_t u_n = \left[\dfrac{\partial u_n}{\partial \underline{n}}\right] & \text{on } K_n, \\ u_n = 0 & \text{on } \partial\Omega, \\ [u_n] = 0 & \text{on } K_n \end{cases} \tag{1}$$

where f is a given function in $L^2(\Omega)$; μ_n is a scaling factor which is only important for the asymptotic analysis of the problems and so we choose it equal to one, since we work with n fixed; Δ_t denotes the Laplace–Beltrami operator on K_n; $[u_n]$ and $[\frac{\partial u_n}{\partial \underline{n}}]$ denote the jump of u_n and of its normal derivative across K_n respectively. Let D be a bounded polygon and $\{v_1, \ldots, v_m\}$ be the vertices of its reentrant corners. We denote by $r_i : D \to \mathbb{R}$ the distance function from the vertex v_i in D, by $\mathcal{I}_i(\varepsilon) := \{x \in D\colon\ r_i(x) < \varepsilon\}$, and by $r : D \to \mathbb{R}$ a smooth weighing function such that $r(x) = r_i(x)$, $x \in \mathcal{I}_i(\varepsilon)$ and $r(x) = 1$, $x \in D \setminus \bigcup_{i=1}^m \mathcal{I}_i(2\varepsilon)$, for a fixed, possibly small, $\varepsilon > 0$. We denote by $H^{2,\alpha}(D)$, $0 < \alpha < 1$, the weighted Sobolev space equipped with the norm $\|\cdot\|^2_{H^{2,\alpha}(D)} = \|\cdot\|^2_{1,D} + \sum_{|\beta|=2} \|r^\alpha D^\beta \cdot\|^2_{0,D}$. In [6], the variational formulation of (1) is provided together with existence, uniqueness and regularity results. We summarize all these results in the following proposition, where we denote by u_n^i the restriction of u_n to Ω_n^i for $i = 1,2$ and by ∇_t the tangential gradient along K_n.

Proposition 2.1. *For every $f \in L^2(\Omega)$, the problem* (1) *is equivalent to the following well-posed variational equation*: *find $u_n \in V(\Omega, K_n) = \{v \in H_0^1(\Omega)\colon\ v|_{K_n} \in H_0^1(K_n)\}$ such that*

$$(\mathrm{P}_n) \quad \int_\Omega \nabla u_n \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}y + \int_{K_n} \nabla_t u_n \cdot \nabla_t v \, \mathrm{d}s = \int_\Omega f v \, \mathrm{d}x \, \mathrm{d}y, \quad \forall v \in V(\Omega, K_n), \tag{2}$$

which has one and only one solution u_n. Moreover u_n satisfies the following inequality

$$\|u_n\|_{V(\Omega,K_n)} \leqslant c\|f\|_{L^2(\Omega)}, \tag{3}$$

for a constant c independent from n and we also have that

- $u_n^i \in H^{2,\alpha_i}(\Omega_n^i)$, $i = 1,2$, $\alpha_1 > \frac{2}{5}$, $\alpha_2 > \frac{1}{4}$,
- $u_n|_{K_n} \in H^2(K_n)$.

We recall that, for all $s \geqslant 1$, $H^s(K_n) = \{u \in H^1(K_n)\colon\ u|_M \in H^s(M),\ M = \text{segment of } K_n\}$ (for definitions and details see [2] and also [6]). We observe that the solution exhibits a singular behavior near to the reentrant corners of the curve K_n and so we construct an appropriate mesh refinement algorithm.

3. The mesh construction

The aim of this work is to formulate a Galerkin approximation of the problem (2). The first step of the finite element method is the construction of an appropriate mesh of the domain, adapted to the physical properties of the problem. The use of optimal meshes, in terms of the local mesh size, node positions and quality of the elements, leads to the

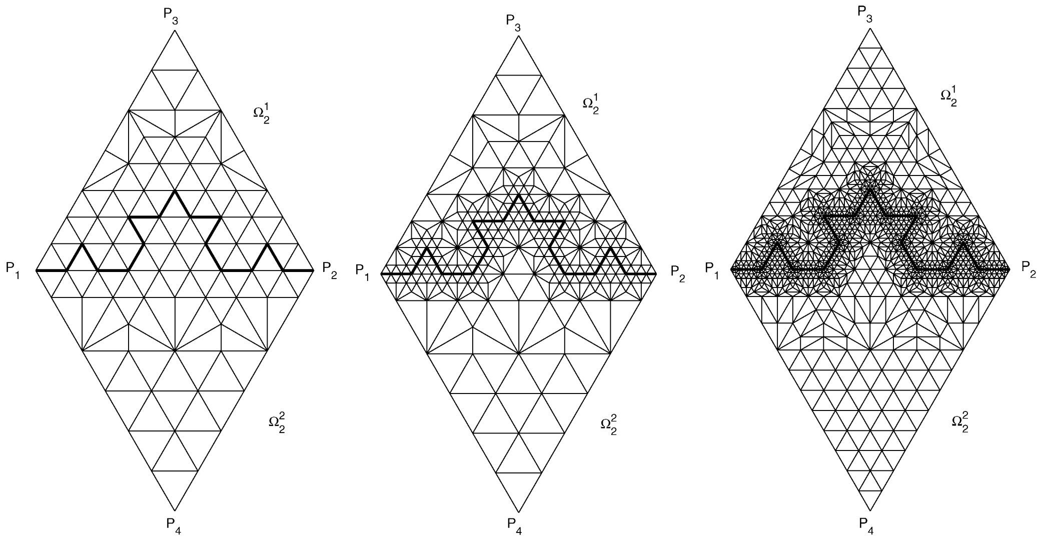

Fig. 1. Left: the mesh adapted to the geometric shape of the domain. Middle: the first (local) refinement. Right: the second (global) refinement.

computation of more accurate discrete solutions. In view of further researches on the asymptotic behavior of discrete solutions (i.e, when $n \to \infty$), we construct a sequence of 'nested meshes', $\{\mathcal{T}_{n,h}\}_{n\in\mathbb{N}}$, one for each approximated problem (P_n), with the following features:

– the vertices of the prefractal curves are nodes of the triangulations;
– the meshes are conformal and they form a regular and non quasi-uniform family.

With the same features mentioned above and for any fixed n in $\mathbb{N}$, we generate a local and a global refinement mesh algorithm by a suitable splitting of the elements according to the intersection between an element T and the prefractal curve K_n is composed either of one edge or two, or of one vertex or two, or it is empty. We provide examples of meshes generated with our set of rules in Fig. 1. Iterating this partitioning of the elements, we generate a sequence of nested meshes $\{\mathcal{T}_{n,h_j}\}_{j\in\mathbb{N}}$, which satisfy the set of assumptions of the following theorem, for $D = \Omega_n^i$ and $\alpha = \alpha_i$, $i = 1, 2$.

Theorem 3.1. *Let D be a non-convex polygonal domain, and $\{\mathcal{T}_h\}_{h>0}$ a regular family of meshes on D. Let h and h_T be the global and local mesh sizes, respectively, and $\{\mathcal{T}_h\}_{h>0}$ be locally refined towards reentrant corners in the following sense: for a fixed $\alpha < 1$, there exists a $\sigma > 0$ such that*

(a) *$h_T \leqslant \sigma h^{1/(1-\alpha)}$ for every $T \in \mathcal{T}_h$ such that a vertex of T is a reentrant corner of D;*
(b) *$h_T \leqslant \sigma h \inf_T r^\alpha$ for every $T \in \mathcal{T}_h$ in a neighborhood of a reentrant corner of D.*

Then, the space $V_h \subset \mathcal{C}^0(\overline{D})$ of piecewise affine polynomials of $\mathcal{T}_h$ verifies:

$$\inf_{v_h\in V_h} |u - v_h|_{1,D} \leqslant Ch \sum_{|\beta|=2} |r^\alpha D^\beta u|_{0,D} \quad \forall u \in H^{2,\alpha}(D). \tag{4}$$

This theorem is due to Babuška et al. (see [1, Theorem 5.1]).

4. Galerkin approximation of the problem

We approximate the problem (P_n) by a conforming Galerkin method (see [3] for the theory of finite elements). From now on we suppose n fixed in $\mathbb{N}$. Let $\{\mathcal{T}_{n,h_j}\}_{j\in\mathbb{N}}$ be a regular family of triangulations of the domain Ω which is constructed as in Section 3 and let $h_j := \max\{\mathrm{diam}(T),\ T \in \mathcal{T}_{n,h_j}\}$ be the size of the j-th triangulation. For all

j in $\mathbb{N}$, we define the finite dimensional space $X^1_{n,h_j} := \{v \in C^\circ(\overline{\Omega}) \text{ s.t. } v|_T \in \mathbb{P}_1,\ \forall T \in \mathcal{T}_{n,h_j}\}$, where $\mathbb{P}_1$ denotes the polynomials of degree 1. By setting $V^1_{n,h_j}(\Omega) := X^1_{n,h_j} \cap H^1_0(\Omega)$, we get $V^1_{n,h_j}(\Omega) \subset V(\Omega, K_n)$. The Galerkin approximation for (P_n) reads: Find $u_{n,h_j} \in V^1_{n,h_j}(\Omega)$ such that

$$(\mathrm{P}_{n,j}) \quad \int_\Omega \nabla u_{n,h_j} \cdot \nabla v_{h_j}\,\mathrm{d}x\,\mathrm{d}y + \int_{K_n} \nabla_t u_{n,h_j} \cdot \nabla_t v_{h_j}\,\mathrm{d}s = \int_\Omega f v_{h_j}\,\mathrm{d}x\,\mathrm{d}y, \quad \forall v_{h_j} \in V^1_{n,h_j}(\Omega) \tag{5}$$

for all j in $\mathbb{N}$. Finally, we formulate our main theorem:

Theorem 4.1. *Let u_n and u_{n,h_j} be the solutions of (P_n) and $(\mathrm{P}_{n,j})$, respectively. The following error estimate holds*:

$$\|u_n - u_{n,h_j}\|_{V(\Omega,K_n)} \leqslant C h_j \left\{ \sum_{i=1,2} \sum_{|\beta|=2} \left| r^{\alpha_i} D^\beta u^i_n \right|^2_{0,\Omega^i_n} + |u_n|^2_{2,K_n} \right\}^{1/2}, \tag{6}$$

where C is a constant independent from h_j and n, for all $j \in \mathbb{N}$.

Proof. It is a consequence of Theorem 3.1 and Cea's Lemma. □

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