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Dynamical Systems

Backward volume contraction for endomorphisms with eventual volume expansion ☆

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Abstract

We consider smooth maps on compact Riemannian manifolds. We prove that under some mild condition of eventual volume expansion Lebesgue almost everywhere we have uniform backward volume contraction on every pre-orbit of Lebesgue almost every point. ***To cite this article: J.F. Alves et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Contraction en arrière pour des endomorphisms en expansion. Nous considérons des transformations différentiables sur des varietés Riemannienes compactes. Nous montrons que dans une certaine condition modérée d'expansion de volume nous pouvons déduire que pour Lebesgue presque chaque point nous avons contraction uniforme de volume en arrière de chaque pré-orbite. ***Pour citer cet article : J.F. Alves et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Statement of results

Let M be a compact Riemannian manifold and let Leb be a volume form on M that we call Lebesgue measure. We take $f: M \to M$ any smooth map. Let $0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$ be a sequence converging to infinity. We define

$$h(x) = \min\{n > 0\colon\ |\det Df^n(x)| \geqslant a_n\}, \tag{1}$$

if this minimum exists, and $h(x) = \infty$, otherwise. For $n \geqslant 1$, we take

$$\Gamma_n = \{x \in M\colon\ h(x) \geqslant n\}. \tag{2}$$

Theorem 1.1. *Assume that $h \in L^p(\text{Leb})$, for some $p > 3$, and take $\gamma < (p-3)/(p-1)$. Choose any sequence $0 < b_1 \leqslant b_2 \leqslant b_3 \leqslant \cdots$ such that $b_k b_n \geqslant b_{k+n}$ for every $k, n \in \mathbb{N}$, and assume that there is $n_0 \in \mathbb{N}$ such that*

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$b_n \leqslant \min\{a_n, \mathrm{Leb}(\Gamma_n)^{-\gamma}\}$ *for every* $n \geqslant n_0$. *Then, for* Leb *almost every* $x \in M$, *there exists* $C_x > 0$ *such that* $|\det Df^n(y)| > C_x b_n$ *for every* $y \in f^{-n}(x)$.

We say that $f : M \to M$ is *eventually volume expanding* if there exists $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\sup_{n\geqslant 1} \frac{1}{n} \log\left|\det Df^n(x)\right| > \lambda. \tag{3}$$

Let h and Γ_n be defined as in (1) and (2), associated to the sequence $a_n = \mathrm{e}^{\lambda n}$.

Corollary 1.2. *Assume that* f *is eventually volume expanding. Given* $\alpha > 0$ *there is* $\beta > 0$ *such that for Lebesgue almost every* $x \in M$ *there are* $C_x > 0$ *such that for every* $n \geqslant 1$ *and any* $y \in f^{-n}(x)$

(i) *if* $\mathrm{Leb}(\Gamma_n) \leqslant \mathcal{O}(\mathrm{e}^{-\alpha n})$, *then* $|\det Df^n(y)| > C_x\, \mathrm{e}^{\beta n}$;
(ii) *if* $\mathrm{Leb}(\Gamma_n) \leqslant \mathcal{O}(\mathrm{e}^{-\alpha n^\tau})$ *for some* $\tau > 0$, *then* $|\det Df^n(y)| > C_x\, \mathrm{e}^{\beta n^\tau}$;
(iii) *if* $\mathrm{Leb}(\Gamma_n) \leqslant \mathcal{O}(n^{-\alpha})$ *and* $\alpha > 4$, *then* $|\det Df^n(y)| > C_x n^\beta$.

Specific rates will be obtained in Section 4 for some eventually volume expanding endomorphisms. In particular, non-uniformly expanding maps such as quadratic maps and Viana maps will be considered.

2. Concatenated collections

Let $(U_n)_n$ be a collection of measurable subsets of M whose union covers a full Lebesgue measure subset of M. We say that $(U_n)_n$ is a *concatenated collection* if:

$$x \in U_n \quad \text{and} \quad f^n(x) \in U_m \quad \Rightarrow \quad x \in U_{n+m}.$$

Given $x \in \bigcup_{n\geqslant 1} U_n$, we define $u(x)$ as the minimum $n \in \mathbb{N}$ for which $x \in U_n$. Note that by definition we have $x \in U_{u(x)}$. We define the *chain generated by* $x \in \bigcup_{n\geqslant 1} U_n$ as $C(x) = \{x, f(x), \ldots, f^{u(x)-1}(x)\}$.

Lemma 2.1. *Let* $(U_n)_n$ *be a concatenated collection. If* $\sum_{n\geqslant 1} \sum_{j=0}^{n-1} \mathrm{Leb}(f^j(u^{-1}(n))) < \infty$, *then we have* $\sup\{u(y){:}\ y \in \bigcup_{n\geqslant 1} U_n$ *and* $x \in C(y)\} < \infty$ *for Lebesgue almost every* $x \in M$.

Assume that for a given $x \in M$ there exists an infinite number of chains $C_j = \{y_j, f(y_j), \ldots, f^{s_j-1}(y_j)\}$, $j \geqslant 1$, containing x with $s_j \to \infty$. For each $j \geqslant 1$ let $1 \leqslant r_j < s_j$ be such that $x = f^{r_j}(y_j)$. First we verify that $\lim r_j = \infty$. If not, then replacing by a subsequence, we may assume that there is $N > 0$ such that $r_j < N$ for every $j \geqslant 1$. This implies that $y_j \in \bigcup_{i=1}^N f^{-i}(x)$ for every $j \geqslant 1$. At this point we need the smoothness of f. By compactness of M, the points x in M such that $\#(f^{-i}(x)) = \infty$ are singular values of f^i, $i \in \mathbb{N}$. By Sard's theorem, the set of singular values of a C^1 map is a zero Lebesgue measure set. So, for almost all $x \in M$ we have $\#(\bigcup_{i=1}^N f^{-i}(x)) < \infty$. As the number of chains is infinite, we obtain a contradiction. Since $r_j \to \infty$ and $x = f^{r_j}(y_j) \in f^{r_j}(u^{-1}(s_j))$, then we have $x \in \bigcup_{n\geqslant k} \bigcup_{j=0}^{n-1} f^j(u^{-1}(n))$, $\forall k \geqslant 1$. The assumption $\sum_{n\geqslant 1} \sum_{j=0}^{n-1} \mathrm{Leb}(f^j(u^{-1}(n))) < \infty$ implies that $\mathrm{Leb}(\bigcup_{n\geqslant k} \bigcup_{j=0}^{n-1} f^j(u^{-1}(n))) \to 0$, when $k \to \infty$. This completes the proof of Lemma 2.1.

Lemma 2.2. *Let* $(U_n)_n$ *be a concatenated collection. If* $\sup\{u(y){:}\ y \in \bigcup_{n\geqslant 1} U_n$ *and* $x \in C(y)\} \leqslant N$ *and* x *is not a periodic point, then* $f^{-n}(x) \subset U_n \cup \cdots \cup U_{n+N}$ *for all* $n \geqslant 1$.

Assume that $\sup\{u(y){:}\ y \in \bigcup_{n\geqslant 1} U_n$ and $x \in C(y)\} \leqslant N$, and take $z \in f^{-n}(x)$. Let $z_j = f^j(z)$ for each $j \geqslant 0$. We distinguish the cases $x \in C(z)$ and $x \notin C(z)$. If $x \in C(z)$, and since x is not a periodic point, then $n \leqslant \#C(z) = u(z)$ by definition of $u(\cdot)$ and $u(z) \leqslant N$, since N is an upper bound for $u(z)$, $x \in C(z)$. Hence $n \leqslant u(z) \leqslant N \leqslant n + N$ and we conclude that $z \in U_{u(z)} \subset U_n \cup \cdots \cup U_{n+N}$. If $x \notin C(z)$, then letting $u_0 = u(z)$ we must have $u_0 < n$. Let $u_1 = u(z_{u_0})$. If $u_0 + u_1 < n$ we take $u_2 = u(z_{u_0+u_1})$. We proceed in this way until we find the first $s \leqslant n$ such that $n \leqslant u_0 + \cdots + u_s$.

Note that $u_s = u(z_{u_0+\cdots+u_{s-1}})$, and by the choice of s we must have $x \in C(z_{u_0+\cdots+u_{s-1}})$. Our assumption implies that $u(z_{u_0+\cdots+u_{s-1}}) \leqslant N$, and so $u_0 + \cdots + u_s \leqslant n + N$. By construction we have

$$z \in U_{u_0},\ f^{u_0}(z) = z_{u_0} \in U_{u_1},\ f^{u_0+u_1}(z) = z_{u_0+u_1} \in U_{u_2},\ \ldots,\ f^{u_0+\cdots+u_{s-1}}(z) = z_{u_0+\cdots+u_{s-1}} \in U_{u_s}.$$

By the definition of a concatenated collection we conclude that $z \in U_{u_0+u_1+\cdots+u_s}$.

3. Proofs of main results

Let us now prove Theorem 1.2. Suppose that $h \in L^p(\mathrm{Leb})$, for some $p > 3$. This implies that $\sum_{n\geqslant 1} n^p \,\mathrm{Leb}(h^{-1}(n)) < \infty$, and so there exists some constant $K > 0$ such that $\mathrm{Leb}(h^{-1}(n)) \leqslant Kn^{-p}$ for every $n \geqslant 1$. Now, taking $0 < \gamma < (p-3)/(p-1)$ we have for some $K' > 0$

$$\sum_{n=1}^{\infty} n\left(\sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big)\right)^{1-\gamma} \leqslant \sum_{n=1}^{\infty} n\big(K'/n^{p-1}\big)^{1-\gamma} < \infty.$$

Defining $U_n = \{x \in M\colon\ |\det Df^n(x)| \geqslant b_n\}$, then we have that $\{U_1, U_2, \ldots\}$ is a concatenated collection with respect to the Lebesgue measure. Moreover, setting $U_n^* = U_n \setminus (U_1 \cup \cdots \cup U_{n-1})$ one observes that $U_n^* \subset \bigcup_{m\geqslant n} h^{-1}(m)$, for otherwise there would be $x \in U_n^* \cap h^{-1}(m)$ with $m < n$, and so $a_m \geqslant b_m > |\det Df^m(x)| \geqslant a_m$, which is not possible. As $|\det Df^j(x)| < b_j$ for every $x \in U_n^*$ and $j < n$, we get $\mathrm{Leb}(f^j(U_n^*)) \leqslant b_j \,\mathrm{Leb}(U_n^*)$ for each $j < n$. Hence

$$\begin{aligned}\sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n-1} \mathrm{Leb}\big(f^j\big(U_n^*\big)\big) &\leqslant \sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n-1} b_j \,\mathrm{Leb}\big(U_n^*\big) \\ &\leqslant \sum_{n=n_0+1}^{\infty} \sum_{j=0}^{n_0-1} b_j \,\mathrm{Leb}\big(U_n^*\big) + \sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \,\mathrm{Leb}\big(U_n^*\big) \\ &\leqslant \sum_{j=0}^{n_0-1} b_j + \sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \,\mathrm{Leb}\big(U_n^*\big).\end{aligned}$$

Now, we just have to check that the last term in the sum above is finite. Indeed,

$$\begin{aligned}\sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \,\mathrm{Leb}\big(U_n^*\big) &\leqslant \sum_{n=n_0+1}^{\infty} \sum_{j=n_0}^{n-1} b_j \sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big) \leqslant \sum_{n=n_0+1}^{\infty} n b_n \sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big) \\ &\leqslant \sum_{n=n_0+1}^{\infty} n\left(\sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big)\right)^{-\gamma} \sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big) \\ &= \sum_{n=n_0+1}^{\infty} n\left(\sum_{k=n}^{\infty} \mathrm{Leb}\big(h^{-1}(k)\big)\right)^{1-\gamma} < \infty.\end{aligned}$$

Using the fact that f is eventually volume expanding we deduce that the set of periodic points of f has zero Lebesgue measure. Otherwise, there would be some n for which $\mathrm{Leb}(\mathrm{Fix}(f^n)) > 0$ and almost every $x \in \mathrm{Fix}(f^n)$ having an expanding direction, by eventual volume expansion. By an implicit theorem function argument we deduce that $\mathrm{Fix}(f^n)$ has zero Lebesgue measure in a neighborhood of x. Applying Lemmas 2.1 and 2.2, we get for each generic point $x \in M$ a positive integer number N_x such that if $y \in f^{-n}(x)$ then $y \in U_{n+s}$ for some $0 \leqslant s \leqslant N_x$. Therefore, $|\det Df^{n+s}(y)| > b_{n+s} \geqslant b_n$. Taking $C_x = K^{-N_x}$, where $K = \sup\{|\det Df(z)| : z \in M\}$, we obtain Theorem 1.1:

$$\big|\det Df^n(y)\big| = \frac{|\det Df^{n+s}(y)|}{|\det Df^s(x)|} > C_x b_n.$$

Now we explain how we use Theorem 1.1 to prove Corollary 1.2. Recall that in Corollary 1.2 we have $a_n = \mathrm{e}^{\lambda n}$ for each $n \in \mathbb{N}$. Assume first that $\mathrm{Leb}(\Gamma_n) \leqslant \mathcal{O}(\mathrm{e}^{-c'n})$ for some $c' > 0$. Then it is possible to choose $c > 0$ such that $b_n = \mathrm{e}^{cn}$, for $n \geqslant n_0$. The other two cases are obtained under similar considerations.

4. Examples: non-uniformly expanding maps

An important class of dynamical systems where we can immediately apply our results is the class of non-uniformly expanding dynamical maps introduced in [2]. As particular examples of this kind of systems we present below one-dimensional quadratic maps and the higher dimensional Viana maps.

Quadratic maps. Let $f_a : [-1,1] \to [-1,1]$ be given by $f_a(x) = 1 - ax^2$, for $0 < a \leqslant 2$. Results in [3,6] give that for a positive Lebesgue measure set of parameters f_a in non-uniformly expanding. Freitas [5] proves that for Benedicks–Carleson parameters there are $C, c > 0$ such that $\mathrm{Leb}(\Gamma_n) \leqslant C\,\mathrm{e}^{-cn}$ for every $n \geqslant 1$. Thus, it follows from Corollary 1.2 that *there exists* $\beta > 0$ *such for Lebesgue almost every* $x \in I$ *there is* $C_x > 0$ *such that* $|(f^n)'(y)| > C_x\,\mathrm{e}^{\beta n}$ *for every* $y \in f^{-n}(x)$.

Viana maps. Let $a_0 \in (1,2)$ be such that the critical point $x = 0$ is pre-periodic for the quadratic map $Q(x) = a_0 - x^2$. Let $S^1 = \mathbb{R}/\mathbb{Z}$ and b: $S^1 \to \mathbb{R}$ given by $b(s) = \sin(2\pi s)$. For fixed small $\alpha > 0$, consider the map $\hat{f}$ from $S^1 \times \mathbb{R}$ into itself given by $\hat{f}(s,x) = (\hat{g}(s), \hat{q}(s,x))$, where $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, and $\hat{g}$ is the uniformly expanding map of S^1 defined by $\hat{g}(s) = ds$ (mod $\mathbb{Z}$) for some integer $d \geqslant 2$. For $\alpha > 0$ small enough there is an interval $I \subset (-2,2)$ for which $\hat{f}(S^1 \times I)$ is contained in the interior of $S^1 \times I$. Thus, any map f sufficiently close to $\hat{f}$ in the C^0 topology has $S^1 \times I$ as a forward invariant region. Moreover, there are $C, c > 0$ such that $\mathrm{Leb}(\Gamma_n) \leqslant C\,\mathrm{e}^{-c\sqrt{n}}$ for every $n \geqslant 1$; see [1,4,7]. Thus, it follows from Corollary 1.2 that *there exists* $\beta > 0$ *such for Lebesgue almost every* $X \in S^1 \times I$ *there is* $C_X > 0$ *such that* $|\det Df^n(Y)| > C_X\,\mathrm{e}^{\beta\sqrt{n}}$ *for every* $Y \in f^{-n}(X)$.

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