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Partial Differential Equations/Calculus of Variations

Image restoration and edge detection by topological asymptotic expansion

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Abstract

A new method for edge detection and image restoration is proposed. This method is based on topological gradient approach. Experimental results obtained on noisy images illustrate the capabilities of this promising method in image processing. ***To cite this article: L.J. Belaid et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Restauration d'images et détection de contours par l'asymptotique topologique. Une nouvelle méthode de restauration d'images avec détection de contours est proposée. Cette méthode est basée sur l'approche du gradient topologique. Des résultats expérimentaux obtenus sur des images bruitées illustrent les possibilités de cette approche prometteuse dans le domaine du traîtement d'images. ***Pour citer cet article : L.J. Belaid et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Dans cette Note, on montre qu'il est possible de résoudre le problème de restauration d'images par des moyens d'optimisation topologique. Il s'agit d'adapter l'approche du gradient topologique utilisée pour la détection de fissures [1]. En effet, une image peut être interprétée comme une fonction constante par morceaux, constituée par des régions homogènes, lesquelles sont séparées par un ensemble de discontinuités représentant les contours de l'image. On associe alors les méthodes diffusives au gradient topologique afin de détecter les contours de l'image. Plus encore, la transformation du cosinus discrète est utilisée comme préconditionneur pour la méthode du gradient conjugué, les résultats numériques sont assez prometteurs, dans la mesure où, le temps de calcul et le nombre d'itérations ont été considérablement diminués.

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On considère v une image bruitée donnée, définie dans un domaine $\Omega \subset \mathbb{R}^2$. En notant par u l'image restaurée, on est amené à résoudre le problème aux dérivées partielles donné par l'Éq. (1). Plus précisemment, dans notre approche, c prend uniquement deux valeurs : c_0 dans la partie homogène de l'image et une petite valeur ϵ sur les contours. D'autre part, afin d'utiliser la méthode du gradient topologique [5], on note par $j(\Omega) = J(u_\Omega)$ la fonction coût à minimiser, où u_Ω est la solution du problème (1), et pour $\rho \geqslant 0$ assez petit, soit $\Omega_\rho = \Omega \backslash \sigma_\rho$ le domaine perturbé obtenu après insertion d'une fissure $\sigma_\rho = x_0 + \rho\sigma(n)$, où $x_0 \in \Omega$, $\sigma(n)$ est une fissure droite, et n est le vecteur unité normal à la fissure. Alors la fonction j admet un développement asymptotique topologique, quand ρ tend vers zéro, donné par le Théorème 3.1. $G(x_0, n)$ est appelé alors le gradient topologique, et est donné dans notre cas par la formule $G(x, n) = \langle M(x)n, n\rangle$, où $M(x)$ est une matrice symétrique donnée par l'Éq. (16). On conclut alors, que pour un x donné, $G(x, n)$ prend sa valeur minimale quand n est le vecteur propre associé à la plus petite valeur propre $\lambda_{\min}$ de M. Cette valeur est considérée comme le gradient topologique associé à l'orientation optimale de la fissure $\sigma_\rho(n)$.

1. Introduction

The goal of topological optimization is to find the optimal decomposition of a given domain in two parts: the optimal design and its complementary. Similarly in image processing, the goal is to split an image in several parts, in particular, in image restoration the detection of edges makes this operation straightforward.

We show in this note that it is possible to solve the image restoration problem using topological optimization tools. The basic idea is to adapt the topological gradient approach used for crack detection [1], in fact an image can be viewed as a piecewise smooth function and edges can be considered as a set of singularities. In our context, diffusive methods are associated to the topological gradient to detect edges for image restoration.

Let v be a given noisy image defined in a domain $\Omega \subset \mathbb{R}^2$ and u the restored image. We recall that a classical way to restore the image u is to solve the following PDE problem

$$\begin{cases} -\operatorname{div}(c\nabla u) + u = v & \text{in } \Omega, \\ \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where c is a small positive constant and n denotes the outward unit normal to $\partial\Omega$. This method is well known to give poor results: it blurs important structures like edges. In order to improve this method, nonlinear isotropic and anisotropic methods were introduced, we can cite here the work of Perona and Malik [6], Catté, Lions, Morel and Coll [3] and more recently Weickert [7] and Aubert [2]. In our topological optimization approach, c takes only two values: c_0 in the smooth part of the image and a small value ϵ on edges. Then classical nonlinear diffusive approaches could be seen as a relaxation of our method that increases the instability of the restoration process.

First, we review in Section 2 the classical way (1) of restoring images: we prove that it is exactly the Tikhonov regularization applied to the inversion of the canonical embedding of H^1 into L^2. The topological gradient method [5] and its application to image restoration is presented in Section 3. Some numerical experiments are discussed in Section 4.

2. A classical approach for image restoration

Let K be the canonical embedding operator defined by

$$\begin{aligned} K : H^1(\Omega)/\mathbb{R} &\longrightarrow L^2(\Omega)/\mathbb{R}, \\ u &\longmapsto Ku = u, \end{aligned} \tag{2}$$

where $H^1(\Omega)/\mathbb{R}$ and $L^2(\Omega)/\mathbb{R}$ are the Hilbert spaces given by

$$H^1(\Omega)/\mathbb{R} = \left\{ u \in H^1(\Omega);\ \int_\Omega u(x)\,\mathrm{d}x = 0 \right\} \quad \text{and} \quad L^2(\Omega)/\mathbb{R} = \left\{ v \in L^2(\Omega);\ \int_\Omega v(x)\,\mathrm{d}x = 0 \right\}, \tag{3}$$

and respectively equipped with the following scalars products

$$(u_1, u_2)_{H^1(\Omega)/\mathbb{R}} = \int_\Omega \nabla u_1 \nabla u_2\,\mathrm{d}x \quad \text{and} \quad (v_1, v_2)_{L^2(\Omega)/\mathbb{R}} = \int_\Omega v_1 v_2\,\mathrm{d}x. \tag{4}$$

In order to solve $Ku = v$ where v is a given function in $L^2(\Omega)$, we consider the following minimization problem

$$\inf_{u \in H^1(\Omega)/\mathbb{R}} \int_\Omega |v - Ku|^2 \, \mathrm{d}x. \tag{5}$$

Or if a minimum u of (5) exists, it necessary satisfies the equation

$$K^* K u = K^* v, \tag{6}$$

where K^* is the adjoint of K. Solving (6) is in general an ill-posed problem. The classical idea is to apply the Tikhonov regularization [4]

$$K^* K u + cu = K^* v, \tag{7}$$

where c is a small constant called the regularization coefficient.

Since the variational formulation associated to problem (7) is given by

$$(K^* K u + cu, w)_{H^1(\Omega)/\mathbb{R}} = (K^* v, w)_{H^1(\Omega)/\mathbb{R}} \quad \forall w \in H^1(\Omega)/\mathbb{R}, \tag{8}$$

then, we deduce the following result.

Theorem 2.1. *Problems* (1) *and* (7) *are equivalent.*

Remark 1. Theorem 2.1 means that problem (1) is exactly the Tikhonov regularization of the inversion of the canonical embedding of H^1 into L^2.

3. Application of the topological asymptotic expansion for edge detection

In this section, we use the topological gradient as a tool for detecting edges for image restoration. First, we recall the principle of the topological asymptotic expansion [5] and [1].

Let Ω be an open bounded domain of $\mathbb{R}^2$ and $j(\Omega) = J(u_\Omega)$ be a cost function to be minimized, where u_Ω is the solution to a given PDE problem defined in Ω. For a small $\rho \geqslant 0$, let $\Omega_\rho = \Omega \backslash \sigma_\rho$ the perturbed domain by the insertion of a crack $\sigma_\rho = x_0 + \rho\sigma(n)$, where $x_0 \in \Omega$, $\sigma(n)$ is a straight crack, and n a unit vector normal to the crack. The topological sensitivity theory provides an asymptotic expansion of j when ρ tends to zero. It takes the general form

$$j(\Omega_\rho) - j(\Omega) = f(\rho) G(x_0, n) + \circ\big(f(\rho)\big), \tag{9}$$

where $f(\rho)$ is an explicit positive function going to zero with ρ and $G(x_0, n)$ is called the topological gradient at point x_0.

For v a given function in $L^2(\Omega)$, we consider the following problem: find $u_\rho \in H^1(\Omega_\rho)$ such that

$$\begin{cases} -\operatorname{div}(c \nabla u_\rho) + u_\rho = v & \text{in } \Omega_\rho, \\ \partial_n u_\rho = 0 & \text{on } \partial\Omega_\rho, \end{cases} \tag{10}$$

where n denotes the outward unit normal to $\partial\Omega_\rho$ and c is a constant function. Edge detection is equivalent to look for a subdomain of Ω where the energy is small. So our goal is to minimize the energy norm outside edges

$$j(\rho) = J(u_\rho) = \int_{\Omega_\rho} \|\nabla u_\rho\|^2. \tag{11}$$

By considering p the solution to the adjoint problem

$$\begin{cases} -\operatorname{div}(c \nabla p) + p = -\partial_u J(u) & \text{in } \Omega, \\ \partial_n p = 0 & \text{on } \partial\Omega, \end{cases} \tag{12}$$

we obtain in the case of a crack $\sigma_\rho(n)$ with boundary condition $\partial_n u = 0$ on $\partial\sigma_\rho(n)$, the following topological asymptotic expansion.

Theorem 3.1. *Function j has the following asymptotic expansion*

$$j(\rho) - j(0) = \rho^2 G(x_0, n) + \circ(\rho^2), \tag{13}$$

with

$$G(x_0, n) = -\pi\big(\nabla u(x_0)\cdot n\big)\big(\nabla p(x_0)\cdot n\big) - \pi\big|\nabla u(x_0)\cdot n\big|^2. \tag{14}$$

The topological gradient could be written as

$$G(x, n) = \big\langle M(x)n, n\big\rangle, \tag{15}$$

where $M(x)$ is the symmetric matrix defined by

$$M(x) = -\pi\,\frac{\nabla u(x)\nabla p(x)^{\mathrm{T}} + \nabla p(x)\nabla u(x)^{\mathrm{T}}}{2} - \pi\nabla u(x)\nabla u(x)^{\mathrm{T}}. \tag{16}$$

For a given x, $G(x, n)$ takes its minimal value when n is the eigenvector associated to the lowest eigenvalue $\lambda_{\min}$ of M. This value will be considered as the topological gradient associated to the optimal orientation of the crack $\sigma_\rho(n)$.

4. Numerical applications

The goal of this section is to prove that the Topological Gradient (TG) approach is able to denoise an image and to preserve features such as edges. First, we give an algorithm that consists in inserting small cracks in regions where the topological gradient $\lambda_{\min}$ is smaller than a given threshold $\alpha < 0$. These regions represent the edges of the image. Note that from a numerical point of view, it is more convenient to simulate cracks by a small value of c. Then, we compare our method with a classical Nonlinear Diffusive (ND) approach [2], with c a decreasing function satisfying $c(0) = 1$ and $\lim_{s\to+\infty} c(s) = 0$. We considered the example which gives $c(|\nabla u|) = \frac{1}{\sqrt{1+|\nabla u|^2}}$. The linear systems obtained from the discretisation of (10) and (12) by finite element method are solved using Gauss Elimination (GE) and the Preconditioned Conjugate Gradient (PCG) methods. The preconditioner used is the Discrete Cosine Transform (DCT). We consider the following algorithm

Algorithm.

(i) Initialization: $c = c_0$.
(ii) Calculation of u_0 and v_0: solutions of the direct (10) and adjoint (12) problems.
(iii) Computation of the 2×2 matrix M and its lowest eigenvalue $\lambda_{\min}$ at each point of the domain.
(iv) Set

$$c_1 = \begin{cases} \varepsilon & \text{if } x\in\Omega \text{ such that } \lambda_{\min} < \alpha < 0,\ \varepsilon > 0,\\ c_0 & \text{elsewhere.}\end{cases} \tag{17}$$

(v) Compute u_1, the solution of problem (10) with $c = c_1$.

To point out the efficiency of our method, we considered a human brain scanner image (250×250 pixels) and added 20% random noise to the original image. We obtain the result of restoration by solving two state equations and one adjoint equation, whereas a ND approach requires around fifty iterations, to get a comparable quality of restoration.

To allow a better comparison from numerical point of view between the TG approach and the ND approach, the graphs of computation times according to the size of the image for both the two approaches using GE and PCG methods, are given in Fig. 2. These computation times are calculated on a PC Pentium 4, 256 Mo DDR, and represented using a logarithmic scale for both X and Y axis. Fig. 2 illustrates the efficiency of the TG approach using the PCG method, in fact according to our algorithm, c is a constant in step 2 and c is equal to a constant except on the edges, as given in step 4. This make the preconditionment by the DCT very efficient in our case and not suitable for the ND approach. Finally, to better illustrate edge detection by topological gradient method, we represent the isovalues of the topological gradient. The numerical results are all represented in Figs. 1 and 2.

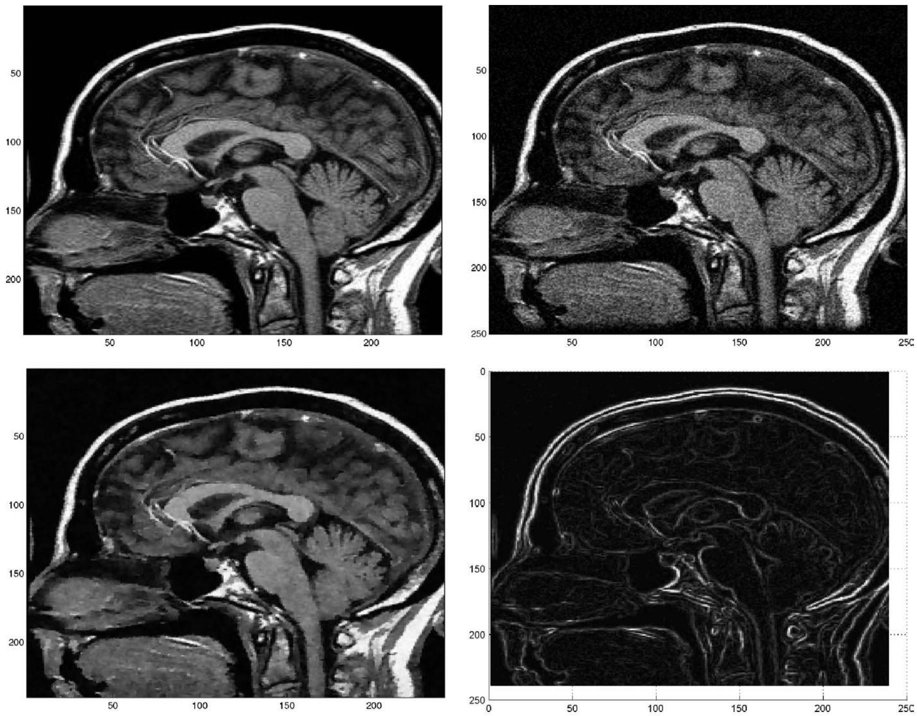

Fig. 1. Top left: initial image, top right: noisy image (20% random noise), bottom left: restored image, bottom right: topological gradient isovalues.

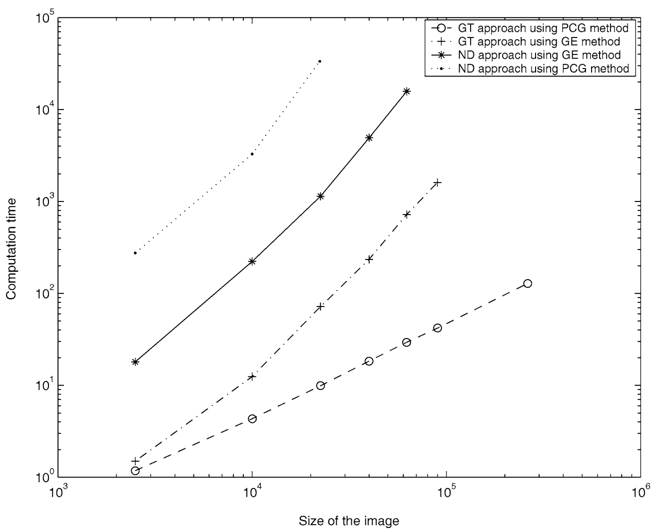

Fig. 2. Variation of the function computation time according to the size of the image using logarithmic scale, for both TG method and ND approaches ($c(t) = \frac{1}{\sqrt{1+t^2}}$) by GE and PCG methods.

5. Conclusion

A new method for image restoration with edge detection has been presented in this work. To make this method relevant with real life applications, we have used the Discrete Cosinus Transform as a preconditioner for the Conjugate Gradient method. The results obtained are very promising especially with computation time and number of iterations. The extension of this method to other problems in image processing such as segmentation and classification, are currently considered.

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