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More compact invariant manifolds appearing in the non-linear coupling of oscillators

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Abstract

Near partially elliptic rest points of generic families of vector fields or transformations, many types of normally hyperbolic invariant compact manifolds can appear, diffeomorphic to intersections of quadrics. ***To cite this article: M. Chaperon et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

D'autres variétés compactes invariantes apparaissant dans le couplage non linéaire d'oscillateurs. Près de points stationnaires partiellement elliptiques de familles génériques de champs de vecteurs ou de transformations apparaissent toutes sortes de variétés compactes invariantes normalement hyperboliques, difféomorphes à des intersections de quadriques. ***Pour citer cet article : M. Chaperon et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Hypothèses. Soit $(u,x)\mapsto X_u(x)\in\mathbf{R}^m$ une famille assez différentiable générique de champs de vecteurs à paramètre $u\in\mathbf{R}^k$, définie au voisinage d'un point de $\mathbf{R}^k\times\mathbf{R}^m$ que l'on peut supposer être 0, telle que $X_0(0)=0$ et que les valeurs propres de $DX_0(0)$ soient imaginaires pures, simples et différentes de 0, d'où $m=2n$, et $k\geqslant n$. On fait l'hypothèse que $k=n+n^2$.

Les valeurs propres de partie imaginaire $\geqslant 0$ étant notées $\mathrm{i}\lambda_1,\dots,\mathrm{i}\lambda_n$, on suppose que, pour $1\leqslant j\leqslant n$, l'équation $\lambda_j=\sum_1^n(p_\ell-q_\ell)\lambda_\ell$ avec $(p,q)\in(\mathbf{N}^n)^2$ et $\sum p_\ell+\sum q_m\leqslant 4$ n'a que les solutions évidentes $p_j=q_j+1$ et $p_\ell=q_\ell$ pour $\ell\neq j$.

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Un changement $(u,x)\mapsto(u,g_u(x))$ de coordonnées locales permet de supposer que $X_u(0)\equiv 0$, que $\mathbf{R}^m=\mathbf{C}^n$, que $L:=DX_0(0)$ est de la forme $Lz=(\mathrm{i}\lambda_1 z_1,\ldots,\mathrm{i}\lambda_n z_n)$ et que $X_u=L+N_u+R_u$, où R_u s'annule à l'ordre 4 en 0, N_u est un champ polynomial (au sens réel) de degré 3 sur $\mathbf{C}^n$ commutant à L, donc invariant par l'action naturelle de $\mathbf{U}(1)^n$, et $N_0=0$. Par conséquent, si l'on pose $r_j:=|z_j|$, la forme normale $L+N_u$ induit sur l'espace des $r=(r_1,\ldots,r_n)$ un champ de vecteurs de la forme $Y_u=\sum_j(a_j+\sum_k b_j^k r_k^2)r_j\frac{\partial}{\partial r_j}$, où les réels a_j, b_j^k sont des fonctions de u.

On fait l'hypothèse que ces fonctions sont nulles en 0 ; la famille étant générique, un changement de coordonnées dans l'espace des paramètres permet donc de supposer que $u=(a_j,(b_j^k)_{1\leqslant k\leqslant n})_{1\leqslant j\leqslant n}$.

Théorème 0.1. *Sous ces hypothèses, quels que soient l'entier impair $q>2$ et les entiers strictement positifs $n_1,\ldots,n_q$ vérifiant $n_1+\cdots+n_q=n$, il existe un ouvert $\Omega_{\mathbf{n}}$ de $\mathbf{R}^k$ adhérent à 0 et tel que, pour tout $u\in\Omega_{\mathbf{n}}$, le champ X_u ait une variété invariante attractive V_u de codimension 3 difféomorphe à la sous-variété $Q_{\mathbf{n}}$ des $\mathbf{C}^n$ formée des $(z_1,\ldots,z_n)$ qui vérifient (1) ci-après, où ρ est une racine primitive q-ième de l'unité et $\|\cdot\|$ la norme euclidienne standard. La sous-variété V_u dépend continûment du paramètre $u\in\Omega_{\mathbf{n}}$ et tend vers $\{0\}$ quand $u\to 0$.*

Remarque 1. Pour $q\geqslant 5$ les $Q_{\mathbf{n}}$ ne sont pas difféomorphes à des produits de sphères comme dans [4] mais à des sommes connexes de produits de deux sphères [5]. Comme dans [4] on peut aussi obtenir des produits attractifs de telles variétés.

Remarque 2. Comme dans [4], l'énoncé correspondant est vrai pour les familles de transformations, et la principale difficulté est de « voir » les V_u sur les formes normales, d'où le nombre élevé de paramètres. Toutes ces variétés invariantes naissent très probablement déjà dans des familles génériques à n paramètres : c'est le cas des sphères invariantes de [4] si $n=2$.

Esquisse de la preuve

Elle repose sur le

Lemme 0.2. *Avec les notations précédentes, on considère les valeurs u_0 suivantes du paramètre u :*

(i) *Tous les a_j sont égaux à un même réel $-\varepsilon a$, avec $\varepsilon>0$.*
(ii) *Pour $1\leqslant \ell, m\leqslant q$, tous les b_j^k avec $n_1+\cdots+n_{\ell-1}<j\leqslant n_1+\cdots+n_\ell$ et $n_1+\cdots+n_{m-1}<k\leqslant n_1+\cdots+n_m$ sont égaux à un même réel c_ℓ^m ; si l'on pose $x_j:=\|v_j\|^2$, la forme normale $L+N_{u_0}$ induit donc sur l'espace des $x=(x_1,\ldots,x_q)$ le champ de vecteurs $Z_{u_0}=2\sum_\ell(-\varepsilon a+\sum_m c_\ell^m x_m)x_\ell\frac{\partial}{\partial x_\ell}$.*
(iii) *La forme normale $L+N_{u_0}$ est tangente à $\sqrt{\varepsilon}\,Q_{\mathbf{n}}$, et Y_{u_0} est* nul *sur l'image de celle-ci ; en d'autres termes, Z_{u_0} est nul sur l'image de cette image, c'est-à-dire qu'il existe pour $1\leqslant\ell\leqslant q$ un unique $c_\ell\in\mathbf{C}$ tel que $-\varepsilon a+\sum_m c_\ell^m x_m=a(\sum_m x_m-\varepsilon)+c_\ell\sum_m\rho^m x_m+\bar{c}_\ell\sum_m\bar{\rho}^m x_m$.*
(iv) *Le champ Z_{u_0} est invariant par permutation circulaire des coordonnées, ce qui se traduit par l'existence de $c\in\mathbf{C}$ tel que $c_\ell=\bar{\rho}^\ell c$ pour $1\leqslant\ell\leqslant q$.*
(v) *On a $a<0$ et $\Re e(c)<0$.*

Alors $\sqrt{\varepsilon}\,Q_{\mathbf{n}}$ est une variété invariante normalement hyperbolique attractive de $L+N_{u_0}$.

Par conséquent, pour ε assez petit et a, c convenables, tous les X_u avec u proche de u_0 ont une variété invariante attractive V_u proche de $\sqrt{\varepsilon}\,Q_{\mathbf{n}}$ et dépendant continûment de u.

1. Introduction

In [4], the secondly-named author proved that many kinds of attracting invariant products of spheres appear in a stable way for arbitrarily small perturbations ('couplings') of systems consisting of n linear oscillators (or quasi-periodic motions). We shall show that other intersections of quadrics, of the type studied in [5,1], can appear in the same situation.

Hypotheses. Let $(u, x) \mapsto X_u(x) \in \mathbf{R}^m$ be a generic smooth enough family of vector fields (resp. diffeomorphisms) with parameter $u \in \mathbf{R}^k$, defined in a neighbourhood of a point of $\mathbf{R}^k \times \mathbf{R}^m$ which we may assume to be 0, such that $X_0(0) = 0$ and that the eigenvalues of $DX_0(0)$ are purely imaginary, simple and different from 0, hence $m = 2n$ and $k \geqslant n$. We assume $k = n + n^2$.

Denoting the eigenvalues with positive imaginary part by $\mathrm{i}\lambda_1, \ldots, \mathrm{i}\lambda_n$, we make the following non-resonance hypothesis: for $1 \leqslant j \leqslant n$, the equation

$$\lambda_j = \sum_1^n (p_\ell - q_\ell)\lambda_\ell$$

has no solution $(p, q) \in (\mathbf{N}^n)^2$ with $\sum p_\ell + \sum q_m \leqslant 4$ other than the obvious ones $p_j = q_j + 1$ and $p_\ell = q_\ell$ for $\ell \neq j$.

By a local change of coordinates $(u, x) \mapsto (u, g_u(x))$, we may assume that $X_u(0) \equiv 0$, that $\mathbf{R}^m = \mathbf{C}^n$, that $L := DX_0(0)$ is of the form $Lz = (\mathrm{i}\lambda_1 z_1, \ldots, \mathrm{i}\lambda_n z_n)$ and that $X_u = L + N_u + R_u$, where R_u vanishes at order 4 at 0, N_u is a (real) polynomial vector field of degree 3 on $\mathbf{C}^n$, commuting with L and therefore invariant by the natural action of $\mathbf{U}(1)^n$, and $N_0 = 0$. Hence, setting $r_j := |z_j|$, the normal form $L + N_u$ induces in the space of $(r_1, \ldots, r_n)$'s a vector field of the form

$$Y_u = \sum_j \bigl(a_j + \sum_k b_j^k r_k^2\bigr) r_j \frac{\partial}{\partial r_j},$$

where the real numbers a_j, b_j^k are functions of u.

We make the hypothesis that these functions vanish at 0; the family being generic, we may perform a coordinate change in parameter space and assume that $u = (a_j, (b_j^k)_{1 \leqslant k \leqslant n})_{1 \leqslant j \leqslant n}$.

Theorem 1.1. *Under those hypotheses, for each choice of an odd integer $q > 2$ and positive integers $n_1, \ldots, n_q$ satisfying $n_1 + \cdots + n_q = n$, there exists an open subset $\Omega_{\mathbf{n}}$ of $\mathbf{R}^k$ adherent to 0 and such that, for every $u \in \Omega_{\mathbf{n}}$, the vector field X_u has an attracting invariant manifold V_u of codimension 3, diffeomorphic to the submanifold $Q_{\mathbf{n}}$ of $\mathbf{C}^n$ consisting of those $(z_1, \ldots, z_n)$ which satisfy*

$$|z_1|^2 + \cdots + |z_n|^2 = 1,$$
$$\rho\|v_1\|^2 + \cdots + \rho^{q-1}\|v_{q-1}\|^2 + \|v_q\|^2 = 0, \quad v_\ell := (z_j)_{n_1+\cdots+n_{\ell-1} < j \leqslant n_1+\cdots+n_\ell}, \tag{1}$$

where ρ is a primitive q-th root of unity and $\|\cdot\|$ denotes the standard Euclidean norm. The submanifold V_u depends continuously on the parameter $u \in \Omega_{\mathbf{n}}$ and tends to $\{0\}$ when $u \to 0$.

Remark 1. By reduction to a center manifold, Theorem 1.1 implies a similar statement in dimension $m \geqslant 2n$, which holds for families with $k \geqslant n + n^2$ parameters containing a generic family with $n + n^2$ parameters.

Remark 2. For $q \geqslant 5$, the $Q_{\mathbf{n}}$'s are not diffeomorphic to products of spheres but to connected sums of products of two spheres The simplest case is when $n = q = 5$, where $Q_{\mathbf{n}}$ is the connected sum of five copies of $\mathbf{S}^3 \times \mathbf{S}^4$. Eq. (1) is a canonical form for a generic intersection of quadrics of the form

$$|z_1|^2 + \cdots + |z_n|^2 = 1; \qquad \mu_1\|z_1\|^2 + \cdots + \mu_n\|z_n\|^2 = 0$$

with $\mu_i \in \mathbf{C}$ [5]. As in [4], we can also get attracting *products* of such manifolds.

The intersections of more such quadrics (i.e., of the above form but with $\mu_i \in \mathbf{C}^k$, $k > 1$) have been studied in [1] where it is shown that they can have a much more complicated topology. No generic canonical form is known in this case, and one cannot expect one that is as symmetric as Eq. (1), so a different construction would be needed to realize them as invariant manifolds.

Remark 3. As in [4], the corresponding statement is true for families of transformations and the main difficulty is to 'see' the V_u's on normal forms, hence the high number of parameters. Most probably, all those invariant manifolds arise already in generic families with n parameters: this is the case for the invariant spheres in [4] if $n = 2$.

2. Idea of the proof

The crucial step is the following lemma:

Lemma 2.1. *Notation being as above, consider the following values u_0 of the parameter u:*

(i) *All the a_j's equal the same real number $-\varepsilon a$, $\varepsilon > 0$.*
(ii) *For $1 \leqslant \ell, m \leqslant q$, all the b_j^k's with $n_1 + \cdots + n_{\ell-1} < j \leqslant n_1 + \cdots + n_\ell$ and $n_1 + \cdots + n_{m-1} < k \leqslant n_1 + \cdots + n_m$ equal the same real number c_ℓ^m; therefore, setting $x_j := \|v_j\|^2$, the normal form $L + N_{u_0}$ induces the vector field $Z_{u_0} = 2\sum_\ell(-\varepsilon a + \sum_m c_\ell^m x_m) x_\ell \frac{\partial}{\partial x_\ell}$ on $\mathbf{R}_+^q$.*
(iii) *The normal form $L + N_{u_0}$ is tangent to $\sqrt{\varepsilon}\, Q_{\mathbf{n}}$, on whose image Y_{u_0}* vanishes identically; *in other words, Z_{u_0} vanishes identically on the image of that image*: *for $1 \leqslant \ell \leqslant q$, there exists a unique $c_\ell \in \mathbf{C}$ such that $-\varepsilon a + \sum_m c_\ell^m x_m = a(\sum_m x_m - \varepsilon) + c_\ell \sum_m \rho^m x_m + \bar{c}_\ell \sum_m \bar{\rho}^m x_m$.*
(iv) *The vector field Z_{u_0} is invariant by circular permutation of the coordinates, which means the existence of $c \in \mathbf{C}$ such that $c_\ell = \bar{\rho}^\ell c$ for $1 \leqslant \ell \leqslant q$.*
(v) *We have $a < 0$ and $\Re(c) < 0$.*

Then, $\sqrt{\varepsilon}\, Q_{\mathbf{n}}$ is an attracting normally hyperbolic invariant manifold of $L + N_{u_0}$.

By standard normal hyperbolicity theory [2,3] (or Theorem 1.3 in [4]), if ε is small enough and a, c convenient, then, for every u close enough to u_0, the vector field X_u admits an attracting invariant manifold close to $\sqrt{\varepsilon}\, Q_{\mathbf{n}}$ and depending continuously on u, hence Theorem 1.1.

Proof of Lemma 2.1. We know that Z_{u_0} vanishes on the simplex S of $\mathbf{R}_+^q$ defined by

$$\begin{aligned} \Delta x &:= x_1 + \cdots + x_q = \varepsilon, \\ \Lambda x &:= \rho x_1 + \cdots + \rho^{q-1} x_{q-1} + x_q = 0, \end{aligned}$$

and that

$$Z_{u_0} = 2\sum_{\ell=1}^{q} \bigl(a(\Delta x - \varepsilon) + c\bar{\rho}^\ell \Lambda x + \bar{c}\rho^\ell \bar{\Lambda} x\bigr) x_\ell \frac{\partial}{\partial x_\ell}. \tag{2}$$

What we wish to show is that, for every $x \in S$, every eigenvalue of the endomorphism of $\mathbf{R}^q / T_x S$ induced by $\mathrm{d}Z_{u_0}(x)$ has a negative real part. By (2), for $x \in S$,

$$\mathrm{d}Z_{u_0}(x) = 2\begin{pmatrix} x_1(a\Delta + c\bar{\rho}\Lambda + \bar{c}\rho\bar{\Lambda}) \\ \vdots \\ x_{q-1}(a\Delta + c\bar{\rho}^{q-1}\Lambda + \bar{c}\rho^{q-1}\bar{\Lambda}) \\ x_q(a\Delta + c\Lambda + \bar{c}\bar{\Lambda}) \end{pmatrix} = 2\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_q \end{pmatrix}\begin{pmatrix} a & \bar{\rho}c & \rho\bar{c} \\ \vdots & \vdots & \vdots \\ a & \bar{\rho}^{q-1}c & \rho^{q-1}\bar{c} \\ a & c & \bar{c} \end{pmatrix}\begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix}$$

and, therefore, the equation $dZ_{u_0}(x)v = 2\lambda v$ implies

$$\begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix}\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_q \end{pmatrix}\begin{pmatrix} a & \bar{\rho}c & \rho\bar{c} \\ \vdots & \vdots & \vdots \\ a & \bar{\rho}^{q-1}c & \rho^{q-1}\bar{c} \\ a & c & \bar{c} \end{pmatrix}\begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix} v = \lambda \begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix} v.$$

It follows that the eigenvalues we are interested in, divided by 2, are the eigenvalues of

$$\begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix}\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_q \end{pmatrix}\begin{pmatrix} a & \bar{\rho}c & \rho\bar{c} \\ \vdots & \vdots & \vdots \\ a & \bar{\rho}^{q-1}c & \rho^{q-1}\bar{c} \\ a & c & \bar{c} \end{pmatrix} = \begin{pmatrix} \Delta \\ \Lambda \\ \bar{\Lambda} \end{pmatrix}\begin{pmatrix} ax_1 & \bar{\rho}cx_1 & \rho\bar{c}x_1 \\ \vdots & \vdots & \vdots \\ ax_{q-1} & \bar{\rho}^{q-1}cx_{q-1} & \rho^{q-1}\bar{c}x_{q-1} \\ ax_q & cx_q & \bar{c}x_q \end{pmatrix}$$

$$= \begin{pmatrix} a\varepsilon & 0 & 0 \\ 0 & c\varepsilon & \bar{c}(\rho^2 x_1 + \cdots + \rho^{2(q-1)} x_{q-1} + x_q) \\ 0 & c(\bar{\rho}^2 x_1 + \cdots + \bar{\rho}^{2(q-1)} x_{q-1} + x_q) & \bar{c}\varepsilon \end{pmatrix},$$

namely $a\varepsilon$ and the eigenvalues of

$$D(x) := \begin{pmatrix} c\varepsilon & \bar{c}(\rho^2 x_1 + \cdots + \rho^{2(q-1)} x_{q-1} + x_q) \\ c(\bar{\rho}^2 x_1 + \cdots + \bar{\rho}^{2(q-1)} x_{q-1} + x_q) & \bar{c}\varepsilon \end{pmatrix};$$

their real parts are all negative if and only if we have

$$a < 0,$$

$\operatorname{tr} D(x) < 0$ and $\det D(x) > 0$; the second condition is equivalent to

$$\Re c < 0,$$

in which case the third is satisfied: indeed, $\det D(x) = |c|^2 \varepsilon^2 (1 - |\varepsilon^{-1}(\rho^2 x_1 + \cdots + \rho^{2(q-1)} x_{q-1} + x_q)|^2)$ and, as $\Delta x = \varepsilon$, we have that $\varepsilon^{-1}(\rho^2 x_1 + \cdots + \rho^{2(q-1)} x_{q-1} + x_q)$ is a convex combination of the q-th roots of unity, which equals none of them since $\{\rho^{2\ell}\}_{1 \leqslant \ell \leqslant q} = \{\rho^j\}_{1 \leqslant j \leqslant q}$ and $\Lambda x = 0$. □

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